The QCD shockwave formalism as an infinite twist TMD framework

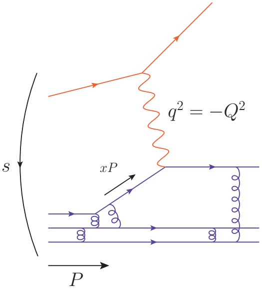
Renaud Boussarie

Brookhaven National Laboratory

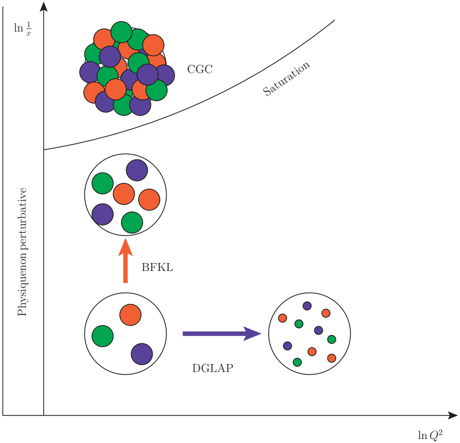
Based on [Altinoluk, RB, Kotko], JHEP 1905 (2019) 156

[Altinoluk, RB], 1902.07930[hep-ph]

Accessing the partonic content of hadrons with an electromagnetic probe

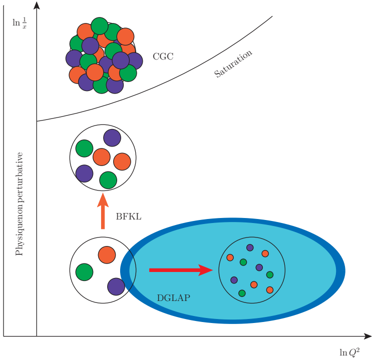


Electron-proton collision (parton model)



QCD at moderate x

$$Q^2 \sim s$$

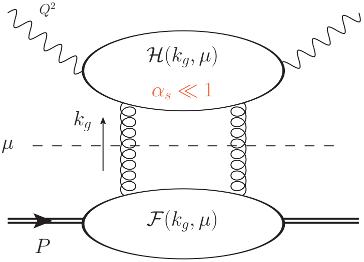


Moderate x QCD: factorization

The process is split into

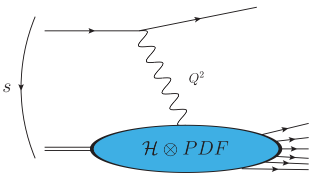
- a hard part $\mathcal{H}$ computed from perturbation theory

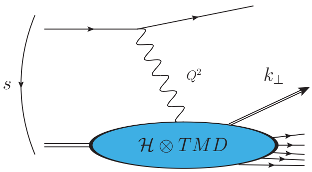
- a parton distribution $\mathcal{F}$ non-perturbative (constrained by experimental data or evaluated with lattice QCD)



- Twist expansion
  $\sigma = \sigma_0 + \frac{1}{Q}\sigma_1 + \ldots$
- Resummation of logarithms
  $\sigma_0 = \sum_n \left[A_n(\alpha_s \ln Q^2)^n + \alpha_s B_n(\alpha_s \ln Q^2)^n \ldots\right]$
- Cancellation of divergences $\Leftrightarrow$ Renormalization of the distribution $\mathcal{F}$
  Ex: DGLAP for a PDF, ERBL for a DA...

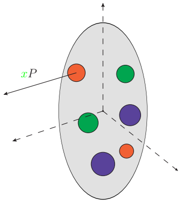
Inclusive processes inclusifs and parton distributions

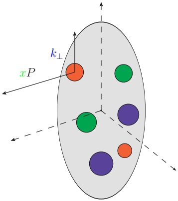




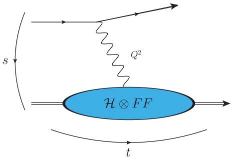
Parton Distribution Fonction (PDF)

Transverse Momentum Dependent distributions (TMD)

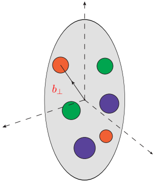


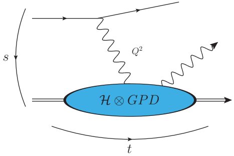


Exclusive processes and parton distributions

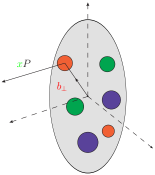


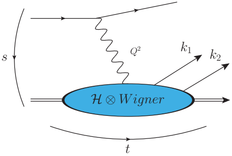
Form Factor



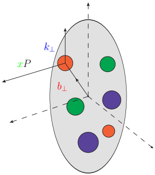


Generalized Parton Distribution (GPD)





Wigner Distribution (GTMD)

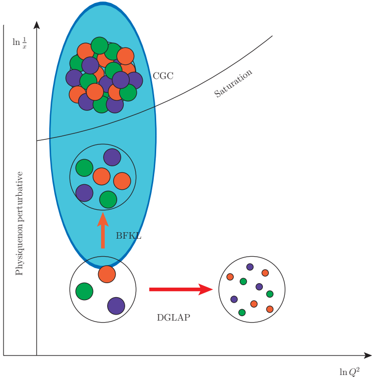


The family tree of parton distributions

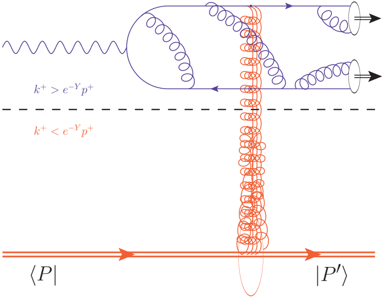
$$
\begin{tikzcd}
 & \begin{array}{c}\text{Wigner distribution}\\ W(x, k_\perp, b_\perp)\end{array} \arrow[dl, "\int d^2 b_\perp"'] \arrow[dr, "\int d^2 k_\perp"] & \\
\begin{array}{c}\text{TMD}\\ f(x, k_\perp)\end{array} \arrow[d, "\int d^2 k_\perp"'] & & \begin{array}{c}\text{GPD}\\ f(x, b_\perp)\end{array} \arrow[dll, "\int d^2 b_\perp"'] \arrow[d, "\int dx"] \\
\begin{array}{c}\text{PDF}\\ f(x)\end{array} \arrow[dr, "\int dx"'] & & \begin{array}{c}\text{Form factor}\\ F(b_\perp)\end{array} \arrow[dl, "\int d^2 b_\perp"] \\
 & \begin{array}{c}\text{Charge}\\ Q\end{array} &
\end{tikzcd}
$$

QCD at small x

$$Q^2 \ll s$$



QCD at small x: QCD shockwaves



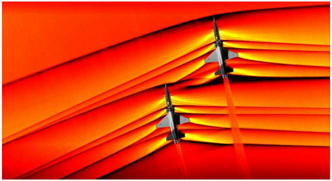
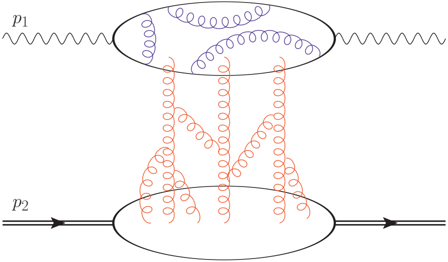


Image : NASA, T-38 planes and shock waves

- Eikonal expansion
  $\sigma = \sigma_0 + \frac{1}{s}\sigma_1 + \ldots$
- Resummation of logarithms
  $\sigma_0 = \sum_n \left[A_n(\alpha_s \ln s)^n + \alpha_s B_n(\alpha_s \ln s)^n \ldots\right]$
- Renormalization group equation for the shockwave operators: (B/JIMWLK) evolution

Kinematics



$$p_1 = p^+ n_1 - \frac{Q^2}{2s} n_2$$

$$p_2 = \frac{m_t^2}{2p_2^-} n_1 + p_2^- n_2$$

$$p^+ \sim p_2^- \sim \sqrt{\frac{s}{2}}$$

Lightcone (Sudakov) vectors

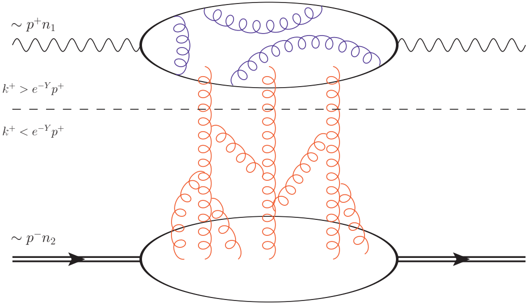
$$n_1 = \sqrt{\frac{1}{2}}(1, 0_\perp, 1), \quad n_2 = \sqrt{\frac{1}{2}}(1, 0_\perp, -1), \quad (n_1 \cdot n_2) = 1$$

Lightcone coordinates:

$$x = (x^0, x^1, x^2, x^3) \to (x^+, x^-, \vec{x})$$

$$x^+ = x_- = (x \cdot n_2) \quad x^- = x_+ = (x \cdot n_1)$$

Rapidity separation

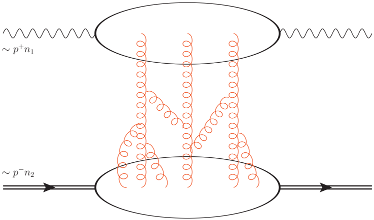


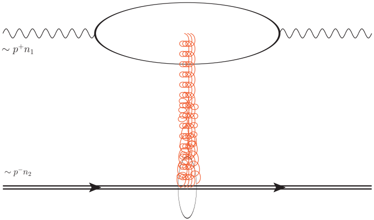
Let us split the gluonic field between "fast" and "slow" gluons

$$\begin{aligned}\mathcal{A}^{\mu a}(k^+, k^-, \vec{k}) &= A^{\mu a}_{Y_c}(|k^+| > e^{-Y_c} p^+, k^-, \vec{k}) \\ &+ b^{\mu a}_{Y_c}(|k^+| < e^{-Y_c} p^+, k^-, \vec{k})\end{aligned}$$

$e^{-Y_c} \ll 1$

Large longitudinal boost to the projectile frame





$$b^+(x^+, x^-, \vec{x}) \qquad \longrightarrow \qquad \frac{1}{\Lambda} b^+\left(\Lambda x^+, \frac{x^-}{\Lambda}, \vec{x}\right)$$

$$b^-(x^+, x^-, \vec{x}) \qquad \qquad \Lambda b^-\left(\Lambda x^+, \frac{x^-}{\Lambda}, \vec{x}\right)$$

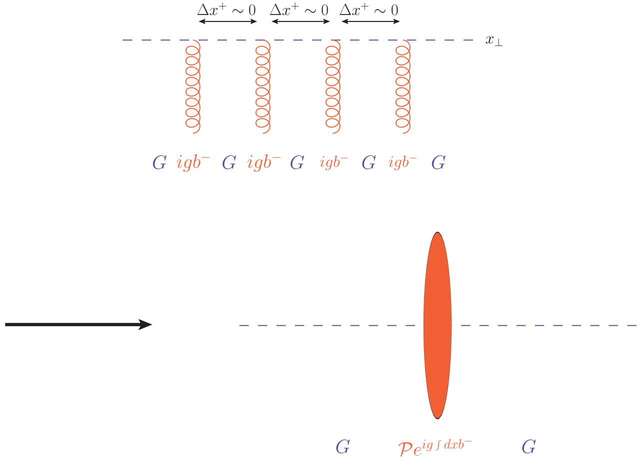
$$b^k(x^+, x^-, \vec{x}) \qquad \Lambda \sim \sqrt{\frac{s}{m_t^2}} \qquad b^k\left(\Lambda x^+, \frac{x^-}{\Lambda}, \vec{x}\right)$$

$$b^\mu(x) \to b^-(x)\, n_2^\mu = \delta(x^+)\, \mathbf{B}(\vec{x})\, n_2^\mu + O\left(\sqrt{\frac{m_t^2}{s}}\right)$$

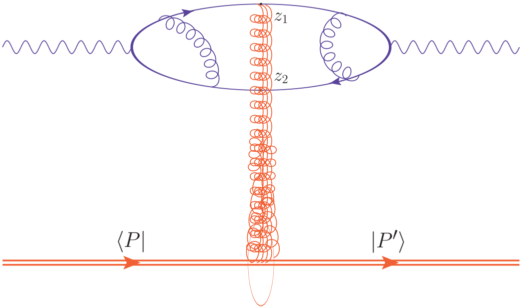
Shockwave approximation

Effective Feynman rules in the slow background field

The interactions with the background field can be exponentiated



Factorized picture



Factorized amplitude

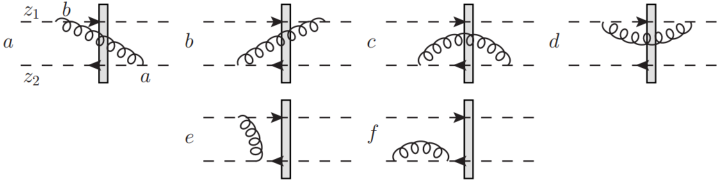
$$\mathcal{A}^{Y_c} = \int d^{D-2}\vec{z}_1 d^{D-2}\vec{z}_2\, \Phi^{Y_c}(\vec{z}_1, \vec{z}_2)\, \langle P'|[\mathrm{Tr}(U^{Y_c}_{\vec{z}_1} U^{Y_c\dagger}_{\vec{z}_2}) - N_c]|P\rangle$$

Dipole operator $\mathcal{U}^{Y_c}_{ij} = \frac{1}{N_c}\mathrm{Tr}(U^{Y_c}_{\vec{z}_i} U^{Y_c\dagger}_{\vec{z}_j}) - 1$

Written similarly for any number of Wilson lines in any color representation!

Evolution for the dipole operator

$$\mathcal{U}_{12}^{Y_c+\delta Y} - \mathcal{U}_{12}^{Y_c}$$



B-JIMWLK hierarchy of equations

[Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner]

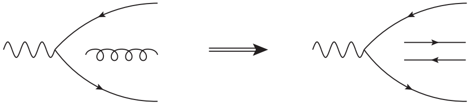
$$\frac{\partial \mathcal{U}_{12}^{Y_c}}{\partial Y_c} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \frac{\vec{z}_{12}^{\,2}}{\vec{z}_{13}^{\,2}\vec{z}_{23}^{\,2}} \left[\mathcal{U}_{13}^{Y_c} + \mathcal{U}_{32}^{Y_c} - \mathcal{U}_{12}^{Y_c} + \mathcal{U}_{13}^{Y_c}\,\mathcal{U}_{32}^{Y_c} \right]$$

$$\frac{\partial \mathcal{U}_{13}^{Y_c}\,\mathcal{U}_{32}^{Y_c}}{\partial Y_c} = \ldots$$

Evolves a dipole into a double dipole

The BK equation

Mean field approximation, or 't Hooft planar limit $N_c \to \infty$ in the dipole B-JIMWLK equation



[Mueller]

$\Rightarrow$ BK equation [Balitsky, 1995] [Kovchegov, 1999]

$$\frac{\partial \left\langle \mathcal{U}_{12}^{Y_c} \right\rangle}{\partial Y_c} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \, \frac{\vec{z}_{12}^{\,2}}{\vec{z}_{13}^{\,2}\vec{z}_{23}^{\,2}} \left[\left\langle \mathcal{U}_{13}^{Y_c} \right\rangle + \left\langle \mathcal{U}_{32}^{Y_c} \right\rangle - \left\langle \mathcal{U}_{12}^{Y_c} \right\rangle + \left\langle \mathcal{U}_{13}^{Y_c} \right\rangle \left\langle \mathcal{U}_{32}^{Y_c} \right\rangle \right]$$

BFKL/BKP part — Triple pomeron vertex

Non-linear term : one type of saturation

Non-perturbative elements are compatible with CGC-type models

Operator product expansion (OPE)

- Moderate x OPE: factorization

$$\mathcal{O}(z) \to \sum_n C_n(z,\mu)\mathcal{O}_n(\mu)$$

  - Operators are ordered in twists (dimension − spin)
  - Divergences in C_n are canceled via renormalization of $\mathcal{O}_n$
  - Easy task: resumming powers of s and logarithms of Q^2. Difficulty: including twist corrections and logarithms of s

- Low x OPE:

$$\mathcal{O}(z) \to C_0(z,Y)\mathcal{O}_0(Y) + \alpha_s C_1(z,Y)\mathcal{O}_1(Y) + \dots$$

  - Operators are sorted by representations of $SU(N_c)$, order by order in α_s
  - Built order by order in α_s. The spurious pole in $C_n(z,Y)$ is canceled via the B/JIMWLK RGE of $\mathcal{O}_{n-1}(Y)$
  - Easy task: resumming twists and logarithms of s. Difficulty: including subeikonal corrections and logarithms of Q^2

The seemingly incompatible nature of the distributions

Two different kinds of gluon distributions

Moderate x distributions	Low x distributions
GTMD, GPD, TMD, PDF...	Dipole scattering amplitude
$\langle P^{(\prime)} \| F^{-i} W F^{-j} W \| P \rangle$	$\langle P^{(\prime)} \| \mathrm{tr}(U_1 U_2^\dagger) \| P \rangle$

(So-called) non-universality of TMD distributions:

The importance of gauge links

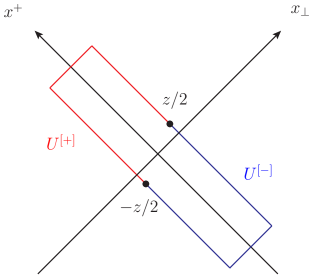
[Collins, Soper, Sterman], [Brodsky, Hwang, Schmidt], [Belitsky, Ji, Yuan], [Bomhof, Mulders, Pijlman], [Boer, Mulders, Pijlman]

[Kharzeev, Kovchegov, Tuchin]

TMD gauge links

"Non-universality" of quark TMD distributions

Gauge links can be future-pointing or past-pointing



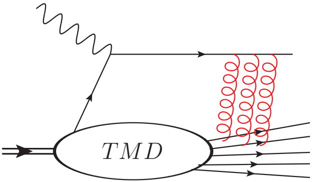
$$q^{[+]}(x, k_\perp) \propto \left\langle P \left| \bar{\psi}\left(\frac{z}{2}\right) \mathcal{U}^{[+]}_{\frac{z}{2},-\frac{z}{2}} \psi\left(-\frac{z}{2}\right) \right| P \right\rangle$$

$$q^{[-]}(x, k_\perp) \propto \left\langle P \left| \bar{\psi}\left(\frac{z}{2}\right) \mathcal{U}^{[-]}_{\frac{z}{2},-\frac{z}{2}} \psi\left(-\frac{z}{2}\right) \right| P \right\rangle$$

For naive T-odd distributions, $q^{[+]} = -q^{[-]}$: Sivers effect

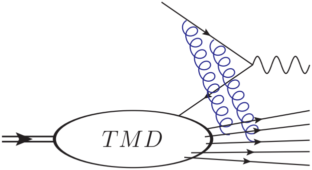
The Sivers effect

SIDIS



Final state interactions: $q^{[+]}$

Drell-Yan

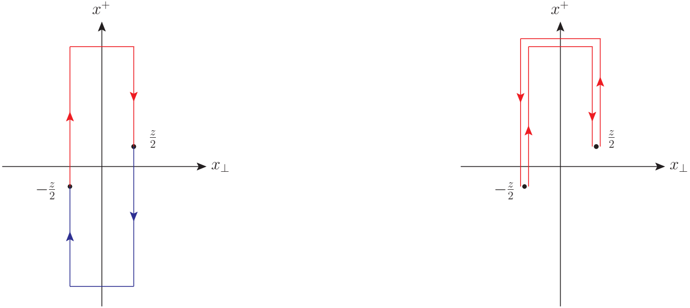


Initial state interactions: $q^{[-]}$

The Sivers distribution comes with a relative $-$ sign between SIDIS and DY: different gauge links for a naive T-odd quantity!

TMD gauge links

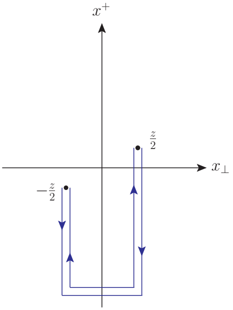
"Non-universality" of gluon TMD distributions



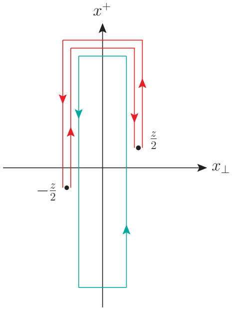
$$\mathrm{Tr}\left[F^{i-}\left(\frac{z}{2}\right)\mathcal{U}^{[-]\dagger}F^{i-}\left(-\frac{z}{2}\right)\mathcal{U}^{[+]}\right] \qquad \mathrm{Tr}\left[F^{i-}\left(\frac{z}{2}\right)\mathcal{U}^{[+]\dagger}F^{i-}\left(-\frac{z}{2}\right)\mathcal{U}^{[+]}\right]$$

TMD gauge links

"Non-universality" of gluon TMD distributions



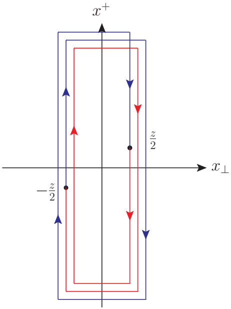
$$\mathrm{Tr}\left[F^{i-}\mathcal{U}^{[-]\dagger}F^{i-}\mathcal{U}^{[-]}\right]$$



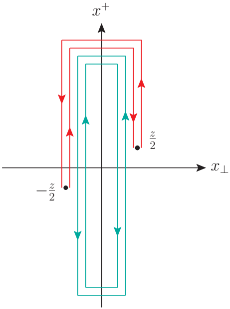
$$\mathrm{Tr}\left[F^{i-}\mathcal{U}^{[+]\dagger}F^{i-}\mathcal{U}^{[+]}\right]\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]$$

TMD gauge links

"Non-universality" of gluon TMD distributions



$$\mathrm{Tr}\left[F^{i-}\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}F^{i-}\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right]$$

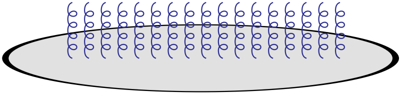


$$\mathrm{Tr}\left[F^{i-}\mathcal{U}^{[+]\dagger}F^{i-}\mathcal{U}^{[+]}\right]\mathrm{Tr}\left[\mathcal{U}^{[\square]}\right]\mathrm{Tr}\left[\mathcal{U}^{[\square]\dagger}\right]$$

TMD distributions from QCD shockwaves

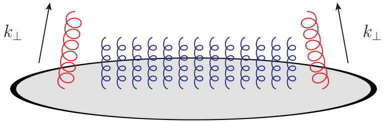
From the CGC to a TMD

From Wilson lines...



$$\left\langle P \left| \mathrm{Tr}\left(U_{\frac{r}{2}} U^{\dagger}_{-\frac{r}{2}} \right) \right| P \right\rangle$$

To a parton distribution



$$\left\langle P \left| \mathrm{Tr}\left(\partial^i U_{\frac{r}{2}} \partial^i U^{\dagger}_{-\frac{r}{2}} \right) \right| P \right\rangle$$

From the CGC to a TMD

Staple gauge links from a Wilson line operator

[Dominguez, Marquet, Xiao, Yuan]

Consider the derivative of a path-ordered Wilson line, denoting

$$[x_1^+, x_2^+]_{\vec{x}} \equiv \mathcal{P} \exp\left[ig \int_{x_1^+}^{x_2^+} dx^+ b^-(x^+, \vec{x}) \right]$$

For a given shockwave operator $U_{\vec{x}} = [-\infty, +\infty]_{\vec{x}}$

$$\partial^i U_{\vec{x}} = ig \int dx^+ [-\infty, x^+]_{\vec{x}} F^{-i}(x^+, \vec{x}) \, [x^+, +\infty]_{\vec{x}}$$

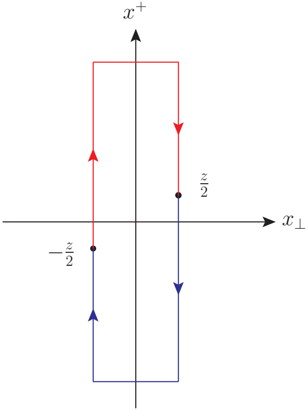
$$\partial^j U_{\vec{x}}^\dagger = -ig \int dx^+ [+\infty, x^+]_{\vec{x}} F^{-j}(x^+, \vec{x}) \, [x^+, -\infty]_{\vec{x}}$$

$$(\partial^i U_{\vec{x}}^\dagger) U_{\vec{x}} = -ig \int dx^+ [+\infty, x^+]_{\vec{x}} F^{-i}(x^+, \vec{x}) \, [x^+, +\infty]_{\vec{x}}$$

Taking the derivative of a shockwave operator allows to extract a physical gluon

From the CGC to a TMD

The dipole TMD

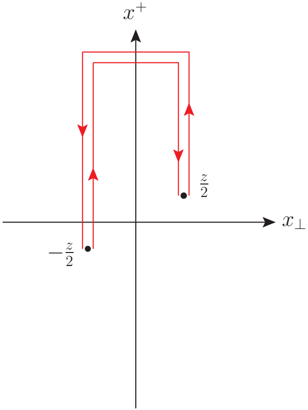


$$\mathcal{F}_{qg}^{(1)}(x,k_\perp) \propto \int d^4z\delta\left(z^+\right)e^{ix(P\cdot z)+i(k_\perp\cdot z_\perp)}\left\langle P\middle|\mathrm{Tr}\left[F^{i-}\left(\frac{z}{2}\right)\mathcal{U}^{[-]\dagger}F^{i-}\left(-\frac{z}{2}\right)\mathcal{U}^{[+]}\right]\middle|P\right\rangle$$

$$\to \int d^2z_\perp e^{i(k_\perp\cdot z_\perp)}\left\langle P\middle|\mathrm{Tr}\left[\left(\partial^i U^\dagger_{\frac{z}{2}}\right)\left(\partial^i U_{-\frac{z}{2}}\right)\right]\middle|P\right\rangle$$

From the CGC to a TMD

The Weizsäcker-Williams TMD



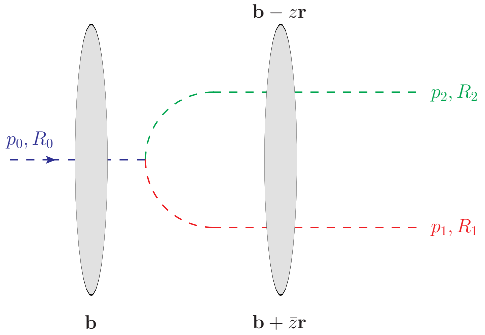
$$\mathcal{F}_{gg}^{(3)}(x,k_\perp) \propto \int d^4z\delta\left(z^+\right)e^{ix(P\cdot z)+i(k_\perp\cdot z_\perp)}\left\langle P\left|\mathrm{Tr}\left[F^{i-}\left(\frac{z}{2}\right)\mathcal{U}^{[+]\dagger}F^{i-}\left(-\frac{z}{2}\right)\mathcal{U}^{[+]}\right]\right|P\right\rangle$$

$$\to \int dz_\perp e^{i(k_\perp\cdot z_\perp)}\left\langle P\left|\mathrm{Tr}\left[\left(\partial^i U_{\frac{z}{2}}\right)U^\dagger_{-\frac{z}{2}}\left(\partial^i U_{-\frac{z}{2}}\right)U^\dagger_{\frac{z}{2}}\right]\right|P\right\rangle$$

General $1 \to 2$ process in the shockwave framework

Splitting of a particle into two particles in the external shockwave field

$$\mathcal{A} = (2\pi)\,\delta\left(p_1^+ + p_2^+ - p_0^+\right) \int d^2\boldsymbol{b}\, d^2\boldsymbol{r}\, e^{-i(\boldsymbol{q}\cdot\boldsymbol{r}) - i(\boldsymbol{k}\cdot\boldsymbol{b})} \mathcal{H}(\boldsymbol{r}) \\ \times \left[\left(U^{R_1}_{\boldsymbol{b}+\bar{z}\boldsymbol{r}}\, T^{R_0}\, U^{R_2}_{\boldsymbol{b}-z\boldsymbol{r}}\right) - \left(U^{R_1}_{\boldsymbol{b}}\, T^{R_0}\, U^{R_2}_{\boldsymbol{b}}\right)\right],$$



Matching shockwave amplitudes and TMD amplitudes

Small dipole "correlation" expansion

Taylor expansion of the Wilson line operators

$$U^{R_1}_{\boldsymbol{b}+\frac{\boldsymbol{r}}{2}} T^{R_0} U^{R_2}_{\boldsymbol{b}-\frac{\boldsymbol{r}}{2}} - U^{R_1}_{\boldsymbol{b}} T^{R_0} U^{R_2}_{\boldsymbol{b}} = \frac{\boldsymbol{r}^i}{2}\left[\left(\partial^i U^{R_1}_{\boldsymbol{b}}\right) T^{R_0} U^{R_2}_{\boldsymbol{b}} - U^{R_1}_{\boldsymbol{b}} T^{R_0}\left(\partial^i U^{R_2}_{\boldsymbol{b}}\right)\right] + O\left(\boldsymbol{r}^2\right)$$

allows for a match at leading twist

$$d\sigma = \mathcal{H}\left(b, r\right) \otimes \left[U^{R_1}_{\boldsymbol{b}+\frac{\boldsymbol{r}}{2}} T^{R_0} U^{R_2}_{\boldsymbol{b}-\frac{\boldsymbol{r}}{2}} - U^{R_1}_{\boldsymbol{b}} T^{R_0} U^{R_2}_{\boldsymbol{b}}\right]$$
$$\times \mathcal{H}^*\left(b', r'\right) \otimes \left[U^{R_2\dagger}_{\boldsymbol{b}'-\frac{\boldsymbol{r}'}{2}} T^{R_0\dagger} U^{R_1\dagger}_{\boldsymbol{b}'+\frac{\boldsymbol{r}'}{2}} - U^{R_2\dagger}_{\boldsymbol{b}'} T^{R_0\dagger} U^{R_1\dagger}_{\boldsymbol{b}'}\right]$$

$$\rightarrow d\sigma^{(i)}_{\boldsymbol{k}=\boldsymbol{0}} \otimes \Phi^{(i)}\left(x, \boldsymbol{k}\right) + O\left(\boldsymbol{r}^2\right)$$

How to extend this to higher twist corrections?

Matching shockwave amplitudes and TMD amplitudes

Power expansion for TMD observables: dealing with powers of $k_\perp/Q$

Consider (hypothetical) hard subamplitudes with non-zero transverse momenta in the t channel. The amplitude would read:

$$\mathcal{H}_1^i(\boldsymbol{k}) \otimes \int d^2\boldsymbol{x}_1 e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}_1)} [\pm\infty, x_1] F^{i-}(x_1) [x_1, \pm\infty]$$

$$+ \mathcal{H}_2^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) \otimes \int d^2\boldsymbol{x}_1 d^2\boldsymbol{x}_2 e^{-i(\boldsymbol{k}_1\cdot\boldsymbol{x}_1)-i(\boldsymbol{k}_2\cdot\boldsymbol{x}_2)} [\pm\infty, x_1] F^{i-}(x_1) [x_1, x_2] F^{j-}(x_2) [x_2, \pm\infty]$$

$$+ \dots$$

$$= \mathcal{H}_1^i(\boldsymbol{k}) \otimes \mathcal{O}_1^i(\boldsymbol{k}) + \mathcal{H}_2^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) \otimes \mathcal{O}_2^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) + \dots$$

Matching shockwave amplitudes and TMD amplitudes

Power expansion for TMD amplitudes:
dealing with powers of $k_\perp / Q$

Leading twist amplitude

$$\mathcal{A}_{LT} = \mathcal{H}_1^i\left(\mathbf{0}\right) \otimes \mathcal{O}_1^i\left(\boldsymbol{k}\right)$$

Next-to-leading twist amplitude

$$\mathcal{A}_{NLT} = \boldsymbol{k} \cdot \left(\partial_{\boldsymbol{k}} \mathcal{H}_1^i\right)\left(\mathbf{0}\right) \otimes \mathcal{O}_1^i\left(\boldsymbol{k}\right) + \mathcal{H}_2^{ij}\left(\mathbf{0}, \mathbf{0}\right) \otimes \mathcal{O}_2^{ij}\left(\boldsymbol{k}_1, \boldsymbol{k}_2\right)$$

First term: kinematic twist correction, second term: genuine twist corrections

Matching shockwave amplitudes and TMD amplitudes

Match without an expansion

Trick: rewrite operators in terms of their derivatives

$$U^{R_1}_{\boldsymbol{b}+\bar{z}\boldsymbol{r}} - U^{R_1}_{\boldsymbol{b}} = -ir^{\mu}_{\perp}\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\int d^2\boldsymbol{b}_1 e^{-i\boldsymbol{k}_1\cdot(\boldsymbol{b}_1-\boldsymbol{b})}\frac{e^{i\bar{z}(\boldsymbol{k}_1\cdot\boldsymbol{r})}-1}{(\boldsymbol{k}_1\cdot\boldsymbol{r})}\left(\partial_\mu U^{R_1}_{\boldsymbol{b}}\right)$$

Rewrite the amplitude

$$\mathcal{A} = (2\pi)\,\delta\left(p_1^+ + p_2^+ - p_0^+\right)\int d^2\boldsymbol{b}\,d^2\boldsymbol{r}\,e^{-i(\boldsymbol{q}\cdot\boldsymbol{r})-i(\boldsymbol{k}\cdot\boldsymbol{b})}\mathcal{H}(\boldsymbol{r})$$
$$\times\left[\left(U^{R_1}_{\boldsymbol{b}+\bar{z}\boldsymbol{r}} - U^{R_1}_{\boldsymbol{b}}\right)T^{R_0}\left(U^{R_2}_{\boldsymbol{b}-z\boldsymbol{r}} - U^{R_2}_{\boldsymbol{b}}\right) + \left(U^{R_1}_{\boldsymbol{b}+\bar{z}\boldsymbol{r}} - U^{R_1}_{\boldsymbol{b}}\right)T^{R_0}U^{R_2}_{\boldsymbol{b}} + U^{R_1}_{\boldsymbol{b}}T^{R_0}\left(U^{R_2}_{\boldsymbol{b}-z\boldsymbol{r}} - U^{R_2}_{\boldsymbol{b}}\right)\right]$$

genuine twist — kinematic + genuine twists

Extracting genuine twists: Taylor, IbP, resummation.

Matching shockwave amplitudes and TMD amplitudes

Finally we can cast the shockwave amplitude into a 1-body amplitude

$$\mathcal{A}_1 = (2\pi)\,\delta\left(p_1^+ + p_2^+ - p_0^+\right)\int d^2\boldsymbol{b}\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{b})}(-i)\int d^2\boldsymbol{r}\, e^{-i(\boldsymbol{q}\cdot\boldsymbol{r})} r_\perp^\alpha \mathcal{H}(\boldsymbol{r})$$

$$\times\left[\left(\frac{e^{i\bar{z}(\boldsymbol{k}\cdot\boldsymbol{r})}-1}{(\boldsymbol{k}\cdot\boldsymbol{r})}\right)\left(\partial_\alpha U_{\boldsymbol{b}}^{R_1}\right) T^{R_0} U_{\boldsymbol{b}}^{R_2} + \left(\frac{e^{-iz(\boldsymbol{k}\cdot\boldsymbol{r})}-1}{(\boldsymbol{k}\cdot\boldsymbol{r})}\right) U_{\boldsymbol{b}}^{R_1} T^{R_0}\left(\partial_\alpha U_{\boldsymbol{b}}^{R_2}\right)\right]$$

and a 2-body amplitude

$$\mathcal{A}_2 = (2\pi)\,\delta\left(p_1^+ + p_2^+ - p_0^+\right)\int \frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}(2\pi)^2\delta^2(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})$$

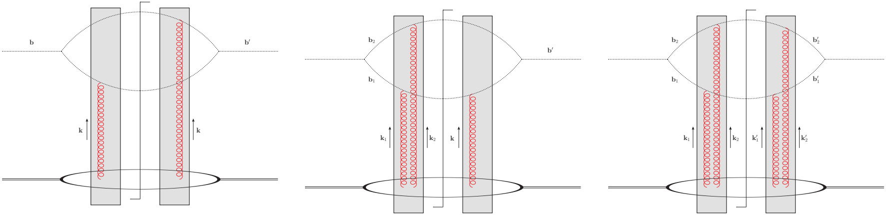
$$\times\int d^2\boldsymbol{b}_1 d^2\boldsymbol{b}_2 e^{-i(\boldsymbol{k}_1\cdot\boldsymbol{b}_1)-i(\boldsymbol{k}_2\cdot\boldsymbol{b}_2)}\left(\partial^i U_{\boldsymbol{b}_1}^{R_1}\right) T^{R_0}\left(\partial^j U_{\boldsymbol{b}_2}^{R_2}\right)$$

$$\times\left[-\int d^2\boldsymbol{r} e^{-i(\boldsymbol{q}\cdot\boldsymbol{r})}\boldsymbol{r}^i\boldsymbol{r}^j\mathcal{H}(\boldsymbol{r})\left(\frac{e^{-iz(\boldsymbol{k}\cdot\boldsymbol{r})}}{(\boldsymbol{k}\cdot\boldsymbol{r})}\frac{e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r})}-1}{(\boldsymbol{k}_1\cdot\boldsymbol{r})} + \frac{e^{i\bar{z}(\boldsymbol{k}\cdot\boldsymbol{r})}}{(\boldsymbol{k}\cdot\boldsymbol{r})}\frac{e^{-i(\boldsymbol{k}_2\cdot\boldsymbol{r})}-1}{(\boldsymbol{k}_2\cdot\boldsymbol{r})}\right)\right]$$

Inclusive low x cross section

Inclusive low x cross section = TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

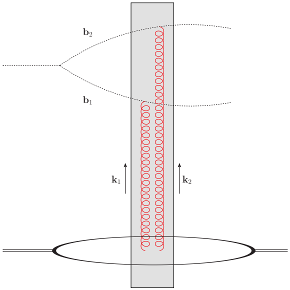


$$\sigma = \mathcal{H}_2^{ij}(k_\perp) \otimes \left\langle P \left| F^{-i} W F^{-j} W \right| P \right\rangle$$
$$+ \mathcal{H}_3^{ijk}(k_\perp, k_{1\perp}) \otimes \left\langle P \left| F^{-i} W g_s F^{-j} W F^{-k} W \right| P \right\rangle$$
$$+ \mathcal{H}_4^{ijkl}(k_\perp, k_{1\perp}, k'_{1\perp}) \otimes \left\langle P \left| F^{-i} W g_s F^{-j} W g_s F^{-k} W F^{-l} W \right| P \right\rangle$$

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude

[Altinoluk, RB]



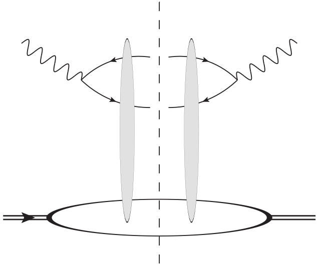
$$\mathcal{H}^{ij}\left(k_{1\perp},\, k_{2\perp}\right) \otimes \left\langle P' \left| F^{-i} W F^{-j} W \right| P \right\rangle$$

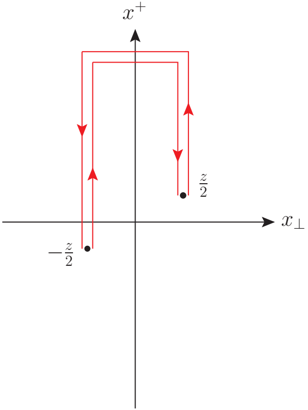
Every exclusive low x process probes a Wigner distribution!

Dijet electro- or photoproduction

Weizsäcker-Williams TMD

$T^{R_0} = 1, U^{R_1} = U, U^{R_2} = U^\dagger$



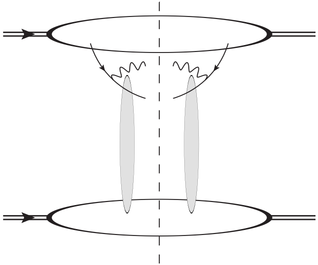


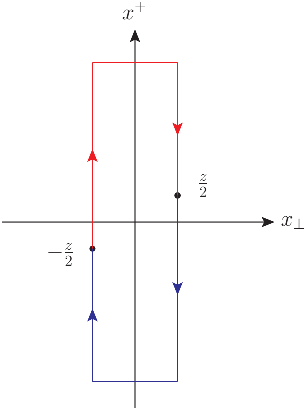
$$\mathcal{F}^{(3)}_{gg}(x \sim 0, k_\perp) \propto \int d^2 z_\perp e^{-i(k_\perp \cdot z_\perp)} \langle P|\mathrm{Tr}(\partial^i U^\dagger_{\frac{z}{2}}) U_{\frac{z}{2}} (\partial^i U^\dagger_{-\frac{z}{2}}) U_{-\frac{z}{2}}|P\rangle$$

Jet+photon production in pA collisions

Dipole TMD

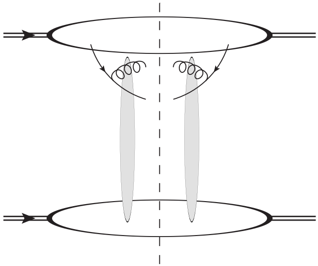
$$T^{R_0} = 1, U^{R_1} = U, U^{R_2} = 1$$

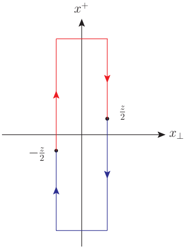




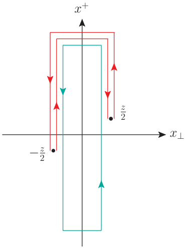
$$\mathcal{F}_{qg}^{(1)}(x \sim 0, k_\perp) \propto \int d^2 z_\perp e^{-i(k_\perp \cdot z_\perp)} \langle P|\mathrm{tr}(\partial^i U_{\frac{z}{2}})(\partial^i U^\dagger_{-\frac{z}{2}})|P\rangle$$

Forward dijet production in *pA* collisions



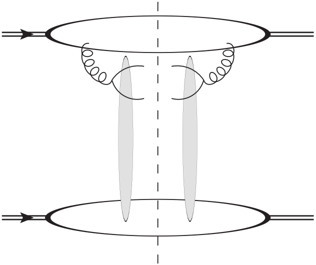


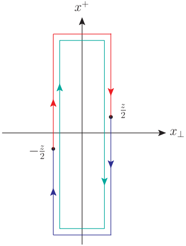
$\mathcal{F}_{qg}^{(1)}$



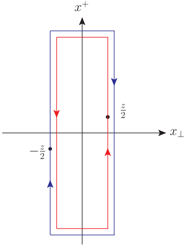
$\mathcal{F}_{qg}^{(2)}$

Forward dijet production in pA collisions

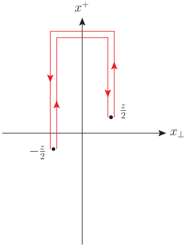




$\mathcal{F}_{gg}^{(1)}$

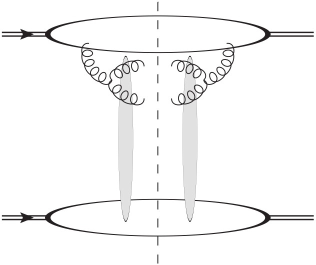


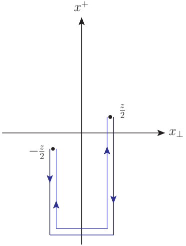
$\mathcal{F}_{gg}^{(2)}$



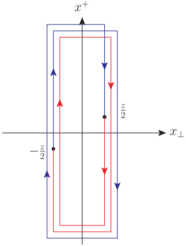
$\mathcal{F}_{gg}^{(3)}$

Forward dijet production in pA collisions

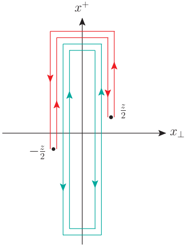




$\mathcal{F}_{gg}^{(4)}$



$\mathcal{F}_{gg}^{(5)}$



$\mathcal{F}_{gg}^{(6)}$

Common tools

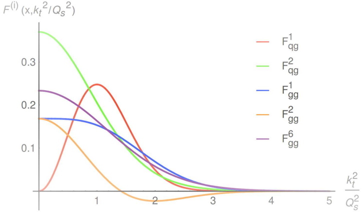
The CGC/TMD equivalence allows to use some TMD tools for the CGC:

- Target Sudakov log resummation for small x processes [Mueller, Xiao, Yuan], [Xiao, Yuan, Zhou]
- Phenomenological Sudakov log simulation [Kotko, Kutak, Sapeta, Stasto, Strikman], [Van Hameren, Kotko, Kutak, Sapeta]

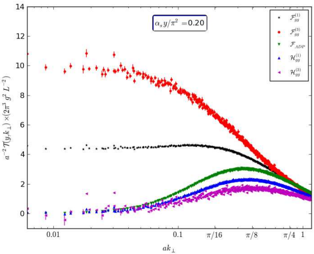
and some CGC tools for small x TMD distributions:

- Golec-Biernat Wüsthoff model for a TMD [Van Hameren, Kotko, Kutak, Marquet, Petreska, Sapeta]
- McLerran-Venugopalan model for a TMD, with JIMWLK evolution [Marquet, Petreska, Roiesnel], [Marquet, Petreska, Taels]

TMD from the CGC



TMD in the GBW model
[Van Hameren, Kotko, Kutak, Marquet, Petreska, Sapeta]



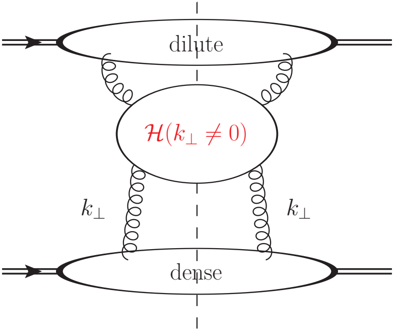
TMD in the MV model with JIMWLK evolution
[Marquet, Petreska, Roiesnel], [Marquet, Roiesnel, Taels]

Saturation in terms of TMD distributions

Small x improved TMD framework (iTMD)

A hybrid framework with off-shell gluons from the target

[Kotko, Kutak, Marquet, Petreska, Sapeta, Van Hameren]



- QCD gauge invariance for multileg amplitudes with an off-shell leg restored with target counterterms [Kotko]
- TMD gauge links are built from the [Bomhof, Mulders, Pijlman] techniques
- Eventually, looks like BFKL, but with distinct TMD distributions for different color flow structures. Interpolates between the TMD regime $|k_\perp| \ll Q$ and the BFKL regime $|k_\perp| \sim Q$

Small-x improved TMD (iTMD) framework
Resums kinematic twists

$$\mathcal{A}_{LT} = \mathcal{H}_1^i\left(\mathbf{0}\right) \otimes \mathcal{O}_1^i\left(\boldsymbol{k}\right) \longmapsto \mathcal{A}_{iTMD} = \mathcal{H}_1^i\left(\boldsymbol{k}\right) \otimes \mathcal{O}_1^i\left(\boldsymbol{k}\right)$$

Neglecting genuine twist corrections (unquantifiable approximation) = Wandzura-Wilczek approximation

iTMD at large k_t: the BFKL limit

- Wandzura-Wilczek approximation
- No multiple scattering from the gauge links

TMD with staple gauge links

$$\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})} \int dx^+ \left\langle P \left| F^{i-}(x) \left[x^+, \pm\infty\right]_{\boldsymbol{x}} \left[\pm\infty, 0^+\right]_{\boldsymbol{0}} F^{j-}(0) \left[0^+, \pm\infty\right]_{\boldsymbol{0}} \left[\pm\infty, x^+\right]_{\boldsymbol{x}} \right| P \right\rangle$$

Large $k_\perp \sim Q \Rightarrow$ small transverse distance $x_\perp$

$$\left[x^+, \pm\infty\right]_{\boldsymbol{x}} \left[\pm\infty, y^+\right]_{\boldsymbol{0}} \sim \left[x^+, y^+\right]_{\boldsymbol{x}\sim\boldsymbol{0}}.$$

All TMD distributions shrink into the unintegrated PDF

$$\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})} \int dx^+ \left\langle P \left| F^{i-}(x) \left[x^+, 0^+\right]_{\boldsymbol{0}} F^{j-}(0) \left[0^+, x^+\right]_{\boldsymbol{0}} \right| P \right\rangle \Big|_{x^-=0}$$

iTMD becomes BFKL

$$d\sigma = \sum_i d\sigma_{\boldsymbol{k}}^{(i)} \otimes \Phi^{(i)}(x, \boldsymbol{k}) = \left(\sum_i d\sigma_{\boldsymbol{k}}^{(i)}\right) \otimes \Phi(x, \boldsymbol{k})$$

BFKL distributions and genuine twist corrections

Unintegrated PDF = 2-Reggeon matrix element

$$\int d^2\boldsymbol{x} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})} \int dx^+ \left\langle P \left| F^{i-}(x)\, [x^+, 0^+]_{\boldsymbol{0}}\, F^{j-}(0)\, [0^+, x^+]_{\boldsymbol{0}} \right| P \right\rangle \Big|_{x^-=0}$$

Integration by parts

$$\int dx^+ \int d^2\boldsymbol{x} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})} \boldsymbol{k}^i \boldsymbol{k}^j \langle P \,|\, [-\infty, x^+]_{\boldsymbol{0}}\, A^-(x)\, [x^+, +\infty]_{\boldsymbol{0}} [+\infty, 0^+]_{\boldsymbol{0}}\, A^-(0)\, [0^+, -\infty]_{\boldsymbol{0}} |\, P \rangle$$

We recognize the so-called nonsense polarizations in axial gauge. We could identify the Reggeon operator:

$$R(x) = \int dx^+ [-\infty, x^+]_{\boldsymbol{0}}\, A^-(x)\, [x^+, +\infty]_{\boldsymbol{0}}$$

and rewrite the unintegrated PDF as

$$\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})} \frac{\boldsymbol{k}^i \boldsymbol{k}^j}{\boldsymbol{k}^2} \boldsymbol{k}^2 \left\langle P \left| R(\boldsymbol{x}) R^\dagger(0) \right| P \right\rangle$$

BFKL distributions and genuine twist corrections

What is missing in BFKL: 3- and 4-Reggeon matrix elements.

$$\langle P\,|RR|\,P\rangle\,,\quad \langle P\,|R(g_sR)R|\,P\rangle\,,\quad \langle P\,|R(g_sR)(g_sR)R|\,P\rangle$$

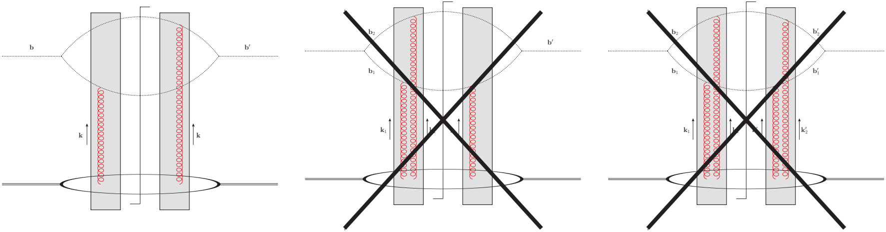
They are not perturbatively suppressed.

Suppression = Wandzura Wilczek approximation (unquantifiable)

Inclusive low x cross section

Inclusive low x cross section $+$ WW $=$ iTMD cross section

[Altinoluk, RB, Kotko]



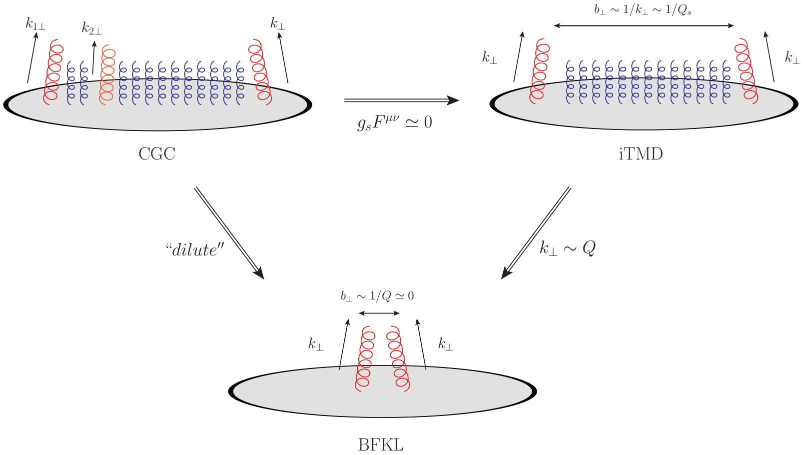
$$\sigma = \mathcal{H}_2^{ij}(k_\perp) \otimes \left\langle P \left| F^{-i} W F^{-j} W \right| P \right\rangle$$

$$+\mathcal{H}_3^{ijk}(k_\perp, k_{1\perp}) \otimes \left\langle P \left| F^{-i} W g_s F^{-j} W F^{-k} W \right| P \right\rangle$$

$$+\mathcal{H}_4^{ijkl}(k_\perp, k_{1\perp}, k'_{1\perp}) \otimes \left\langle P \left| F^{-i} W g_s F^{-j} W g_s F^{-k} W F^{-l} W \right| P \right\rangle$$

The dilute limit

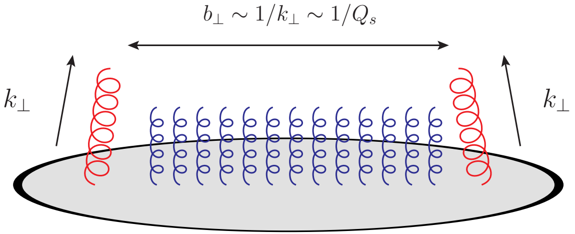
The dilute limit in terms of TMD distributions



Two kinds of multiple scattering effects: higher genuine twists and higher kinematic twists

Kinematic saturation

"Saturation" from a TMD gauge link



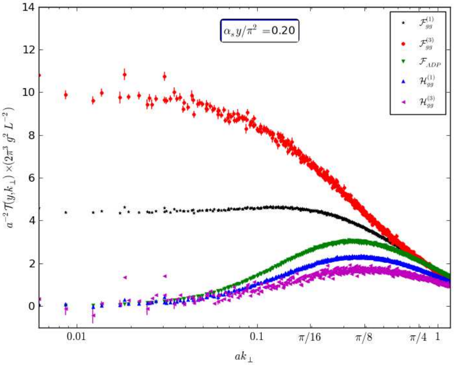
$$g_s^2 \int d^4 b\, \delta\left(b^-\right) e^{i(k\cdot b)} \left\langle P \left| F^{i-}(b)\, \mathcal{U}^{[\pm]}_{b,0} F^{j-}(0)\, \mathcal{U}^{[\pm]}_{0,b} \right| P \right\rangle$$

Expected at small $k_\perp/Q$

Kinematic saturation

"Saturation" from a TMD gauge link

Link length $\sim 1/|k_\perp|$, hence effect suppressed at large $k_\perp$

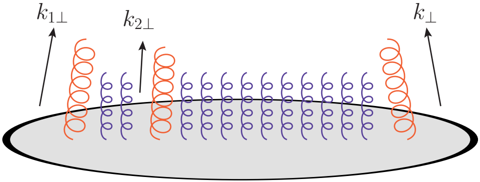


[Marquet, Petreska, Roiesnel ; Marquet, Roiesnel, Taels]

Genuine saturation

"Saturation" as an enhancement of genuine twists

Large gluon occupancy $\Rightarrow g_s F \sim 1$



$$g_s^2 \int d^4 b_1 d^4 b_2 d^4 b' \delta\left(b_1^-\right) \delta\left(b_2^-\right) \delta\left(b'^-\right) e^{i(k_1 \cdot b_1)+i(k_2 \cdot b_2)-i\left(k \cdot b'\right)}$$

$$\times \frac{\left\langle P \left| F^{i-}\left(b_1\right) \mathcal{U}_{b_1, b_2}^{[\pm]} g_s F^{j-}\left(b_2\right) \mathcal{U}_{b_2, b'}^{[\pm]} F^{k-}\left(b'\right) \mathcal{U}_{b', b_1}^{[\pm]} \right| P \right\rangle}{\langle P | P \rangle}$$

$k_\perp / Q$-suppressed: expected at large $k_\perp$?

Linearly polarized gluons at small x

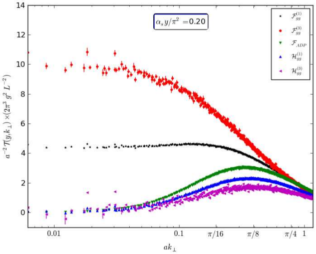
Polarized TMD in the CGC

Wilson line operators also contain linearly polarized gluon TMDs

$$\left\langle P \left| \partial^i U \partial^j U \right| P \right\rangle \to \frac{\delta^{ij}}{2} \mathcal{F}\left(k_\perp\right) + \left(\frac{k_\perp^i k_\perp^j}{k_\perp^2} - \frac{\delta^{ij}}{2} \right) \mathcal{H}\left(k_\perp\right)$$

- They can be computed in the MV model [Metz, Zhou]
- They can be observed in processes with massive quarks [Marquet, Roiesnel, Taels]
- Or in processes with 3 body final states (requires an extension of the notion of the correlation limit) [Altinoluk, RB, Marquet, Taels]
- Can also be seen from loop corrections to 2-body observables, for example prompt photon+jet production in pA collisions [Benić, Dumitru], based on a computation by [Benić, Fukushima, Garcia-Montero, Venugopalan]

Polarized TMD in the CGC



In the large $k_\perp \sim Q$ limit (BFKL limit), all TMDs are equal:

$$\mathcal{F}(k_\perp) = \mathcal{H}(k_\perp), \text{ then } \left\langle P \left| \partial^i U \partial^j U \right| P \right\rangle \to \frac{k_\perp^i k_\perp^j}{k_\perp^2} \mathcal{F}(k_\perp)$$

We can recognize the so-called *non-sense polarization* in lightcone gauge: $\frac{k_\perp^i}{|k_\perp|}$.
BFKL contains as many linearly polarized gluon pairs as unpolarized ones.

At large $k_\perp$, gluon distributions are very polarized

Conclusions

- TMD distributions are what allows to match standard parton distributions and semi-classical descriptions of small x physics
- Dipole and Color Glass Condensate models can give insights on TMDs at small x
- The reformulation of shockwaves in terms of TMD distributions allows to understand polarized gluon distributions at small x
- Two distinct kinds of multiple scattering effects must be distinguished to understand gluonic saturation models