Calculating hadronic tensor on the lattice

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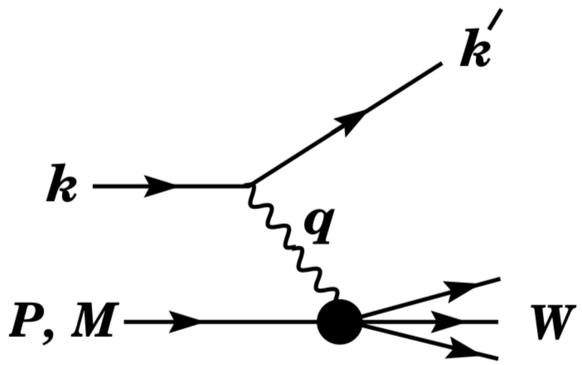
χQCD collaboration

04/01/2019 theory seminar@JLAB

Hadronic tensor



deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)

to leading order perturbation $$\frac{d^2\sigma}{dxdy} = \frac{2\pi y\alpha^2}{Q^4}\sum_j \eta_j L_j^{\mu\nu} W_{\mu\nu}^j$$

the hadronic tensor $$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s \left| \left[J_\mu^\dagger(z)J_\nu(0)\right] \right| p,s \right\rangle$$

for unpolarized cases $$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P\cdot q}F_2(x,Q^2)$$

- ✦ for high energy scatterings (DIS), extract PDFs through factorization

$$F_i = \sum_a C_i^a \otimes f_a$$

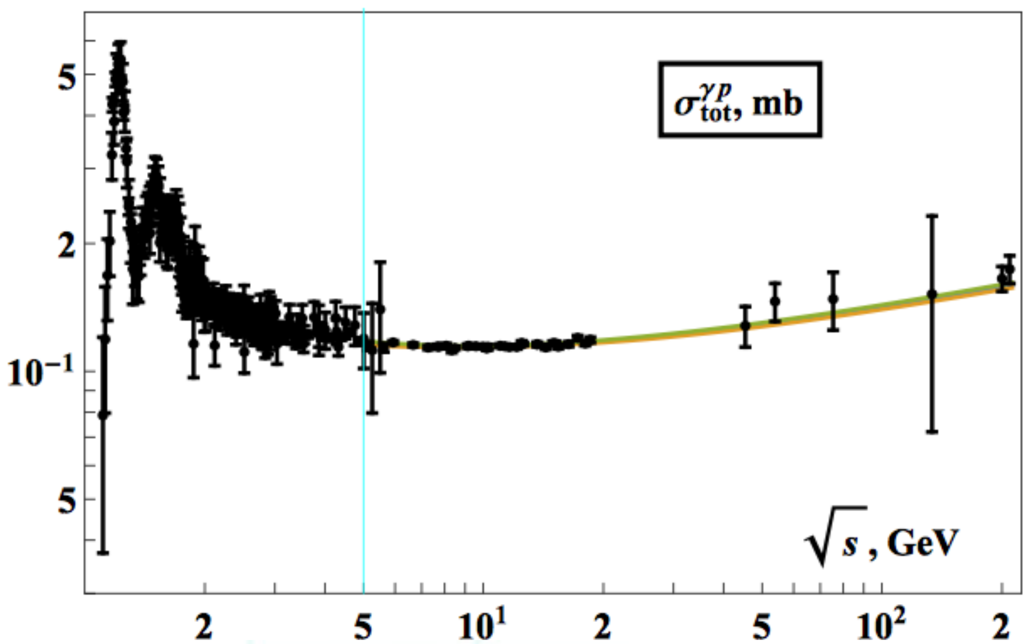
- ✦ for low energy cases (e.g., elastic scatterings), extract form factors

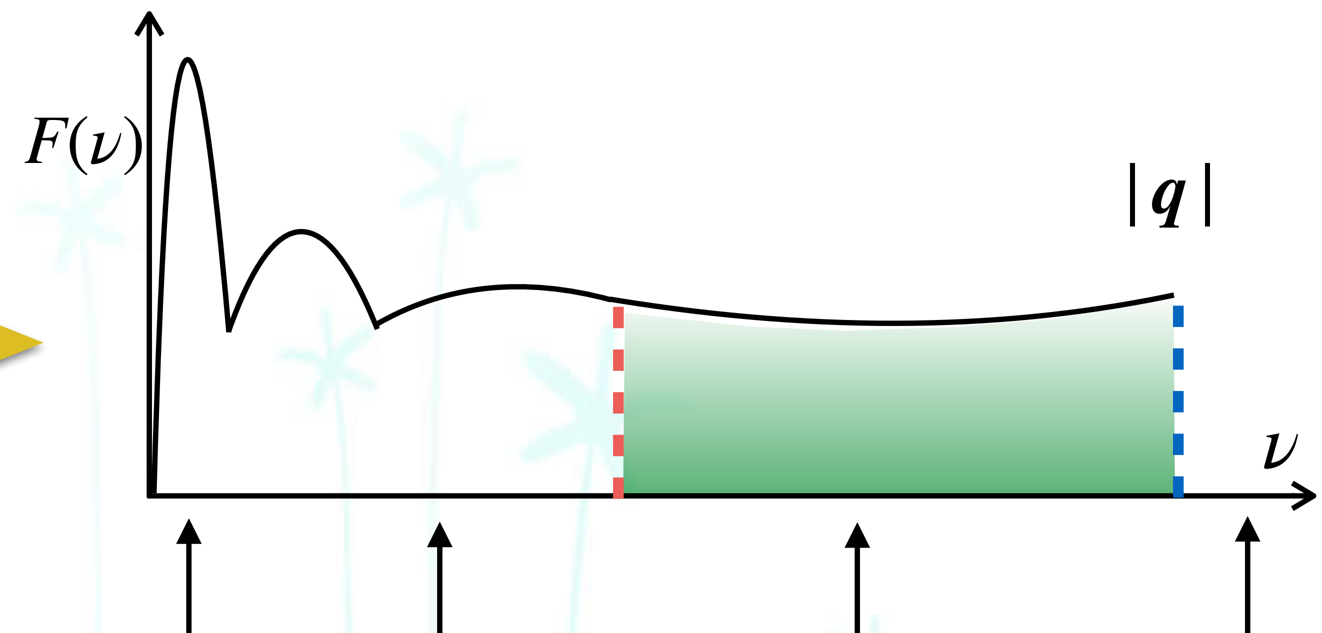
$$F_2^{\mathrm{el}} = \delta(q^2 + 2m_N\nu)\frac{2m_N}{1 - q^2/4m_N{}^2}\left(G_E^2(q^2) - \frac{q^2}{4M_N^2}G_M^2(q^2)\right)$$

Sketch the structure function

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s \left| \left[J^\dagger_\mu(z)J_\nu(0)\right] \right| s,s \right\rangle$$

$$= \frac{1}{4\pi}\sum_n \int \prod_i^n \left[\frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\right] \langle p,s|J^\dagger_\mu(z)|n\rangle\langle n|J_\nu(0)|p,s\rangle(2\pi)^3\delta^4(q-p_n+p)$$





Elastic

nucleon form factors

RES and SIS

N\pi, \Delta, …, continuous spectrum

DIS region

away from the SIS region
$\nu < |\boldsymbol{q}|$ (physical x and Q^2)

$Q^2 < 0$

Motivation 1: parton physics

Lattice efforts:

- ✦ Quasi-PDFs and LaMET
- ✦ Compton amplitude
- ✦ Puedo-PDFs
- ✦ Lattice cross sections
- ✦ …

X. Ji, PRL110, 262002 (2013)

H.W. Lin et. al., PRL121, 242003 (2018)

A. J. Chambers et. al., PRL118, 242001(2017)

A. V. Radyushkin, PRD96, 034025 (2017)

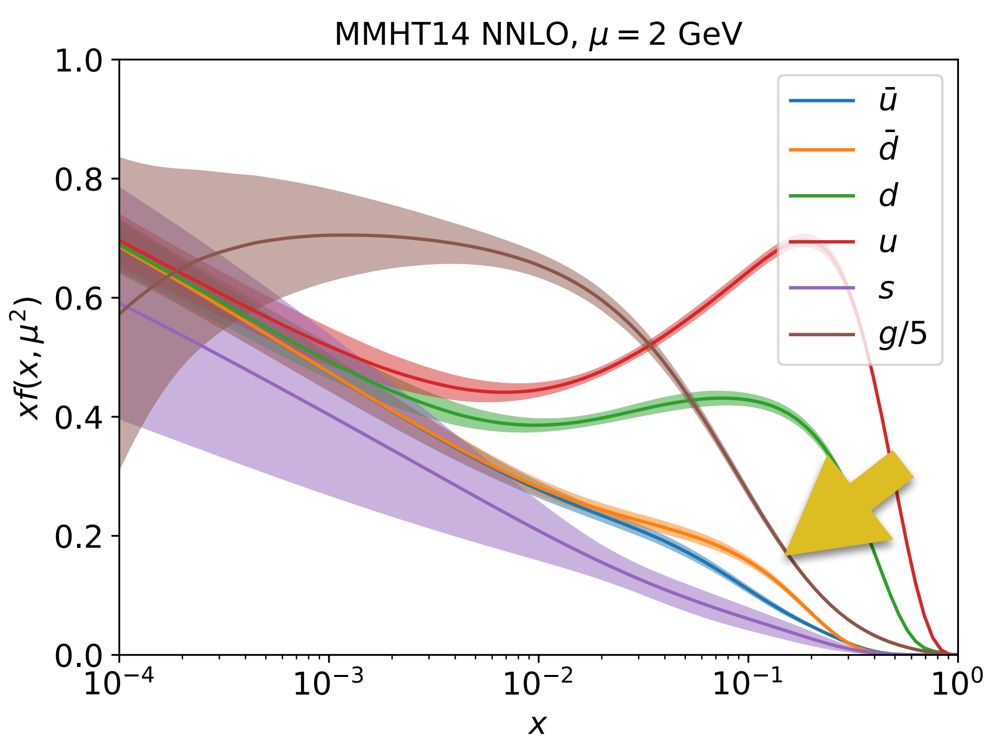
K. Orginos et. al., PRD96, 094503 (2017)

Y.-Q. Ma and J.-W. Qiu, PRL120, 022003 (2018)

R. S. Sufian et. al., arXiv:1901.03921

✦ **Hadronic tensor is scale independent! No need to do renormalization.**

✦ **Structure functions are frame independent! No need of large external momentum.**



L. A. Harland-Lang et. al., EPJ C75, 204 (2015)

The u-bar and d-bar difference of PDFs (Gottfried sum rule violation) is related to the connected-sea anti-partons.

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

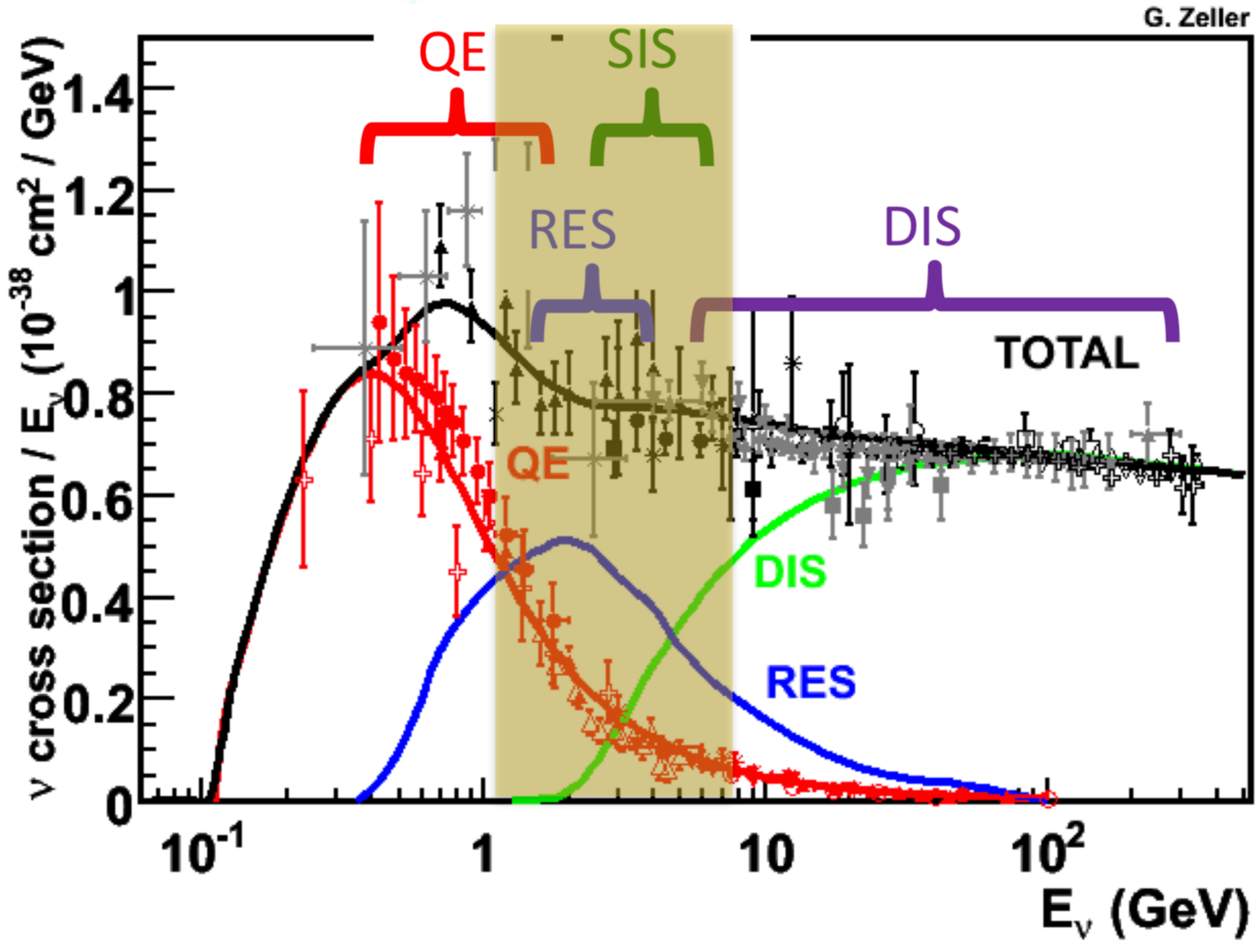
Hadronic tensor provides a direct way to reveal the connected-sea anti-parton contribution.

K.-F. Liu, PRD62, 074501 (2000)

K.-F. Liu, PoS LATTICE2015, 115 (2016)

J. Liang et. al., EPJ Web Conf. 175, 14014 (2018)

Motivation 2: neutrino-nucleus scattering



J.A. Formaggio and G.P. Zeller, RMP84, 1307 (2012); Teppei Katori's talk

- one of the most important tasks in High Energy Physics is to understand the properties of neutrinos.
- DUNE@LBNF FERMILAB with neutrino energy ~1-~7 GeV

- $\nu A \to \nu N$, theoretical input about nucleon structure is needed to help map out the original neutrino beam energy and flux.
- For elastic contribution, nucleon FFs can be calculated by lattice or models.
- But soon enough, one will not be able to tell one state from another and will need INCLUSIVE hadron tensor (the resonance and shallow inelastic scattering (SIS) region).
- The only way that lattice QCD can help as far as we know.

Lattice QCD

Euclidean field theory using the path-integral formalism,

$$t \to -i\tau \qquad \langle O \rangle = \frac{1}{Z}\int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} O e^{-S}$$

Euclidean time correlation functions:

$$C_2(t) = \mathrm{Tr}\left[\Gamma\langle O(t)\bar{O}(0)\rangle\right] = \sum_n \left|\langle 0|O|n\rangle\right|^2 e^{-E_n t} \qquad O(t) = e^{\hat{H}t}O(0)e^{-\hat{H}t}$$

$$C_3(t,\tau) = \mathrm{Tr}\left[\Gamma\langle O(t)C(\tau)\bar{O}(0)\rangle\right] = \sum_{mn} \langle 0|O|n\rangle\langle n|C|m\rangle\langle m|O|0\rangle e^{-E_n(t-\tau)}e^{-E_m\tau}$$

time dependent matrix element can be problematic (e.g., light-cone PDFs)

Minkowski $$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s\left|\left[J_\mu^\dagger(z)J_\nu(0)\right]\right|s,s\right\rangle$$

$$= \frac{1}{2}\sum_n \int \prod_i^n \left[\frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\right]\langle p,s|J_\mu^\dagger(0)|n\rangle\langle n|J_\nu(0)|p,s\rangle(2\pi)^3\delta^4(q-p_n+p)$$

Euclidean $$W'_{\mu\nu} = \frac{1}{4\pi}\sum_n \int dt e^{\left(\nu-(E_n-E_p)\right)t}\int d^3\mathbf{z}e^{i\mathbf{q}\cdot\mathbf{z}}\langle p,s|J_\mu^\dagger(\mathbf{z})|n\rangle\langle n|J_\nu(0)|p,s\rangle$$

$$= \frac{1}{4\pi}\sum_n \boxed{\frac{e^{\left(\nu-(E_n-E_p)\right)T}-1}{\nu-(E_n-E_p)}}\int d^3\mathbf{z}e^{i\mathbf{q}\cdot\mathbf{z}}\langle p,s|J_\mu^\dagger(\mathbf{z})|n\rangle\langle n|J_\nu(0)|p,s\rangle$$

Hadronic tensor on the lattice

four-point function with two currents

$$C_4 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \sum_{\boldsymbol{x}_2 \boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu^\dagger(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

nucleon two-point function

$$C_2 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \left\langle \chi_N(\boldsymbol{x_f}, t_f) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

Euclidean hadronic tensor

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \frac{E_p}{m_N} \frac{\mathrm{Tr}[\Gamma_e C_4]}{\mathrm{Tr}[\Gamma_e C_2]} \to \sum_{\boldsymbol{x}_2 \boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \langle p, s \,|\, J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \,|\, p, s \rangle$$

$$= \sum_n A_n e^{-(E_n - E_p)\tau}, \ \tau \equiv t_2 - t_1$$

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

K.-F. Liu, PRD62, 074501 (2000)

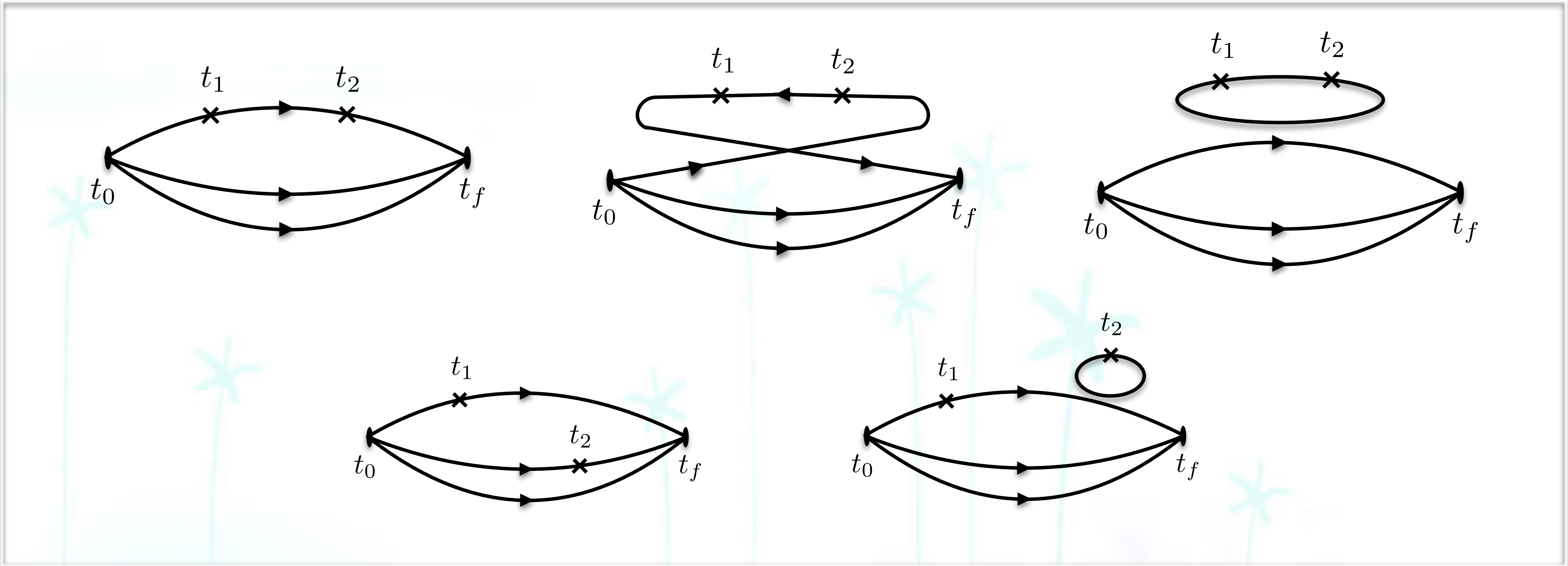
J. Liang et. al., EPJ Web Conf. 175, 14014 (2018)

exponential behavior w.r.t. the time difference between the two currents

Contractions

$$C_4 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \sum_{\boldsymbol{x}_2\boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C\gamma_5 d\right] u_2$$



More contractions if we consider different types of the two currents: vector, axial vector, neutral or charged, various quark flavors …

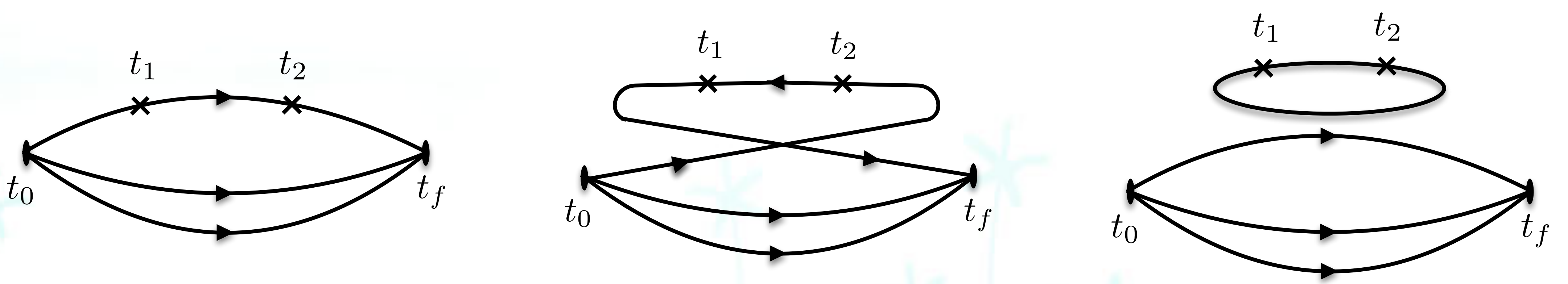
No disconnected insertions in the current plan

The latter two are suppressed when the momentum and energy transfers are large.

Contractions

$$C_4 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \sum_{\boldsymbol{x}_2 \boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C\gamma_5 d\right] u_2$$



valence and connected-sea parton

connected-sea anti-parton (Gottfried sum rule violation)

disconnected-sea parton and anti-parton

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

For the moment, we will focus on the first one.

Back to the Minkowski space

Euclidean hadronic tensor

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) \sim \sum_n A_n e^{-\nu_n \tau}, \nu_n \equiv E_n - E_p$$

Formally, an inverse Laplace transform will do

$$W_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \nu) = \frac{1}{i} \int_{c-i\infty}^{c+i\infty} d\tau e^{\nu\tau} \tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau)$$

Practically, need to solve the inverse problem of the Laplace transform

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \nu) e^{-\nu\tau}$$

↓ **several (O(10)) discrete data points** ↓ **continuous function w.r.t. \nu**

↓

lack of information, an ill-posed problem

More about the inverse problem

A general form

$$c(\tau) = \int k(\tau, \nu)\omega(\nu)d\nu$$

where it is hard to have the inverse function. In our case, the Laplace kernel

$$k(\tau, \nu) = e^{-\nu\tau}$$

$\omega(\nu)$ is a continuous function, but numerically we can discretize it.

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

If the number of nu is less than the number of tau, chi square fitting

If the number of nu is equal to the number of tau, linear equations

If the number of nu is larger than the number of tau, no unique solution

plug in Bayesian prior information?

Inverse problems are everywhere

- Extracting spectral functions from lattice data
- Global fittings of PDFs
- Lattice calculation of Quasi-PDFs

$$\tilde{q}_\Gamma(x, P_z, \tilde{\mu}) = \int_{-\infty}^{\infty} \frac{dz}{2\pi}\, e^{ixP_z z} \langle P | O_\Gamma(z) | P \rangle$$

$$\tilde{q}(x, P_z, p_z^R, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|}\, C\left(\frac{x}{y}, r, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}\right) q(y, \mu)$$

H.-W. Lin et.al., PRL121, 242003(2018)

- Lattice calculation of Pseudo-PDFs

$$\mathfrak{M}_R(\nu, \mu^2) \equiv \int_0^1 dx\, \cos(\nu x)\, q_v(x, \mu^2)$$

K. Orginos et al., PRD96, 094503 (2017)

- Lattice cross sections

$$\sigma_n(\omega, \xi^2, P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x}\, f_a(x, \mu^2) \times K_n^a(x\omega, \xi^2, x^2P^2, \mu^2) + O(\xi^2 \Lambda_{\rm QCD}^2)$$

Y.-Q. Ma and J.-W. Qiu, PRL120, 022003 (2018)

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau,\nu)\omega(\nu)\Delta\nu$$

✦ **Backus-Gilbert (BG)**

G. Backus and F. Gilbert, Geophysical Journal International 16, 169 (1968)

If the kernels can span a complete function basis

$$\sum_{\tau} c(\tau,\nu_0)k(\tau,\nu) \sim \delta(\nu-\nu_0)$$

$$\sum_{\tau} c(\tau,\nu_0)c(\tau) \sim \int \delta(\nu-\nu_0)\omega(\nu)d\nu = \omega(\nu_0)$$

The actual incompleteness of the kernels leads to bad resolution.

✦ **Maximum Entropy (ME)**

E Rietsch et. al., JOURNAL OF GEOPHYSICS, 42:489 (1977)

M. Asakawa et. al., Prog. Part. Nucl. Phys. 46, 459 (2001)

$$P[\omega\,|\,D,\alpha,m] \propto \frac{1}{Z_S Z_L} e^{Q(\omega)} \qquad Q = \alpha S - L$$

$$S = \sum_{\nu}\left[\omega(\nu) - m(\nu) - \omega(\nu)\log\left(\frac{\omega(\nu)}{m(\nu)}\right)\right]\Delta\nu$$

Maximum search is using SVD in a reduced parameter space ($O(10^1)$).

Hyper parameter alpha is averaged over based on assumptions.

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

✦ **Bayesian Reconstruction (BR)**

Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

$$P[\omega | D, \alpha, m] \propto e^{Q'(\omega)} \quad Q' = \alpha S - L - \gamma(L - N_\tau)^2$$

No over fitting

$$S = \sum_{\nu} \left[1 - \frac{\omega(\nu)}{m(\nu)} + \log\left(\frac{\omega(\nu)}{m(\nu)}\right) \right] \Delta\nu$$

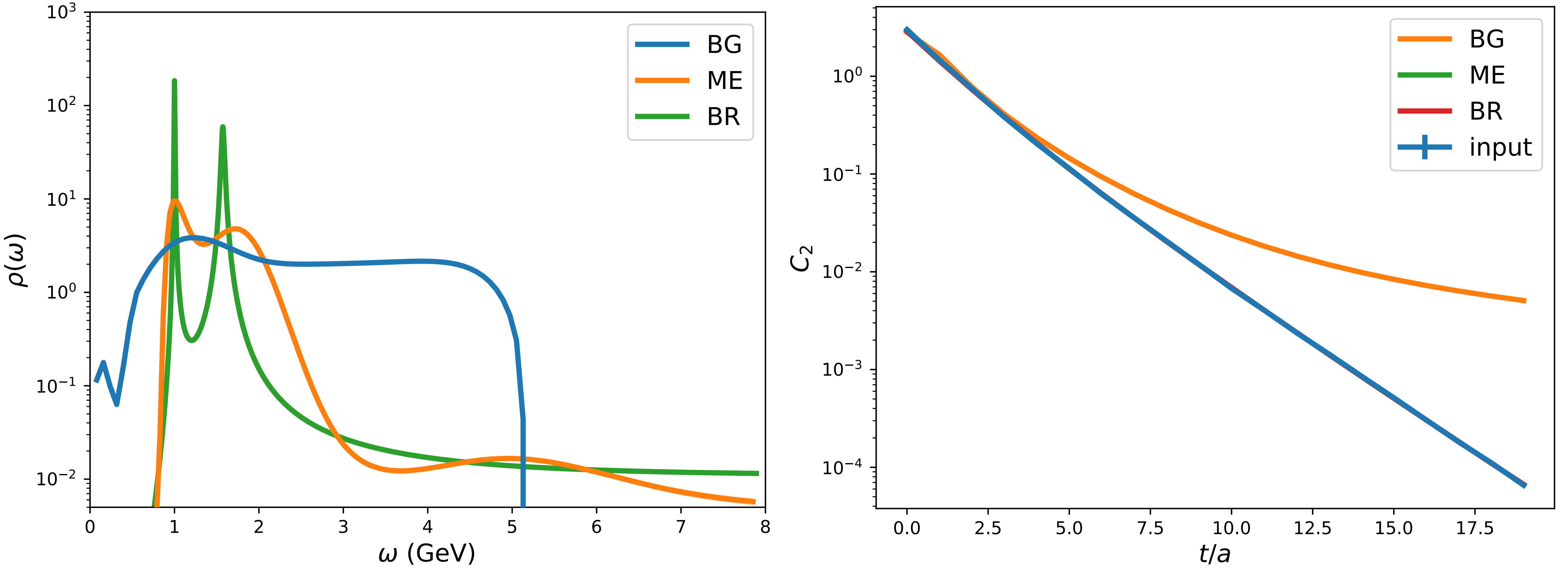
$$P[\omega | D, m] = \frac{P[D | \omega, I]}{P[D | m]} \int d\alpha P[\alpha | D, m]$$

Hyper parameter alpha is integrated over.

Maximum search is in the entire parameter space ($O(10^3)$).

High precision architecture is needed (e.g., 512-bit floating point number).

Tests on nucleon two-point functions

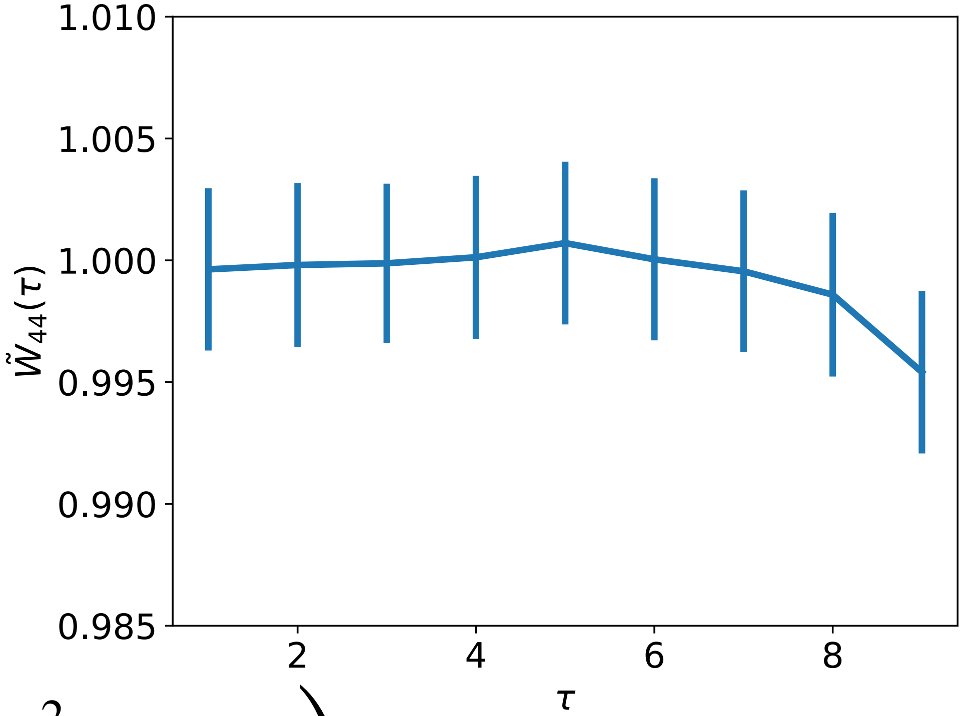


- **mock two-point function data: three single exponentials with mass 1.0, 1.5 and 1.8 GeV respectively, a~0.1 fm, Nt=20, S/N=100**
- **expecting peaks at ~1 GeV and ~1.6 GeV**
- **bad resolution of BG**
- **BR is shaper and more stable than ME**

The elastic case

normalized vector current $J_4 = \bar{\psi}\gamma_4\psi$

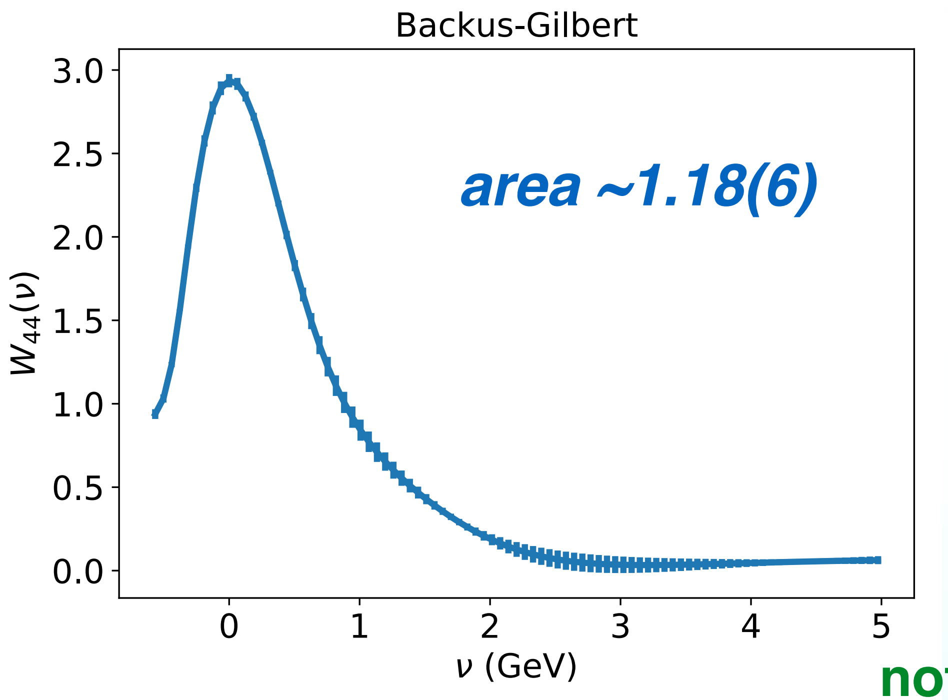
$$\tilde{W}_{44}(\boldsymbol{p}=0,\, \boldsymbol{q}=0,\, \tau) \overset{\tau\to\infty}{=} |\langle N|J_4|N\rangle|^2 e^{-(M_p-M_p)\tau}$$
$$= F_1^2(q^2=0) = g_V^2 = 1$$

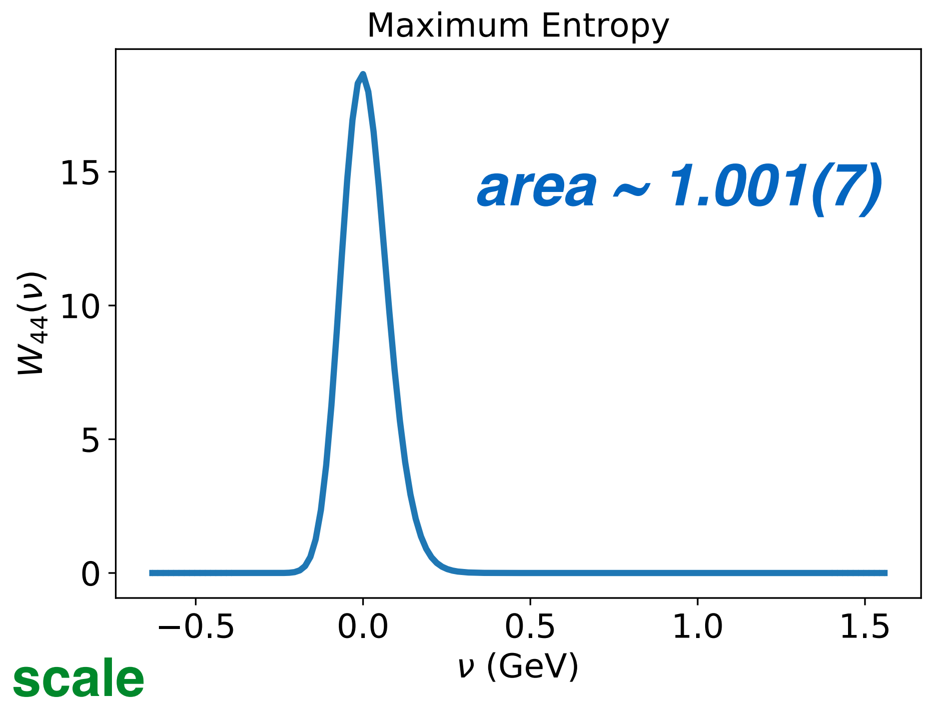


inverse $\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu)e^{-\nu\tau}$

$$W_{44}(q^2,\, \nu) = \delta(q^2 + 2m_N\nu)\frac{2m_N}{1 - q^2/4{m_N}^2}\left(G_E^2(q^2) - \frac{q^2}{4M_N^2}G_M^2(q^2)\right)$$

$$\overset{q^2=0}{=} \delta\nu G_E^2(q^2=0) = \delta\nu g_V^2 = \delta\nu$$ **delta function at zero**





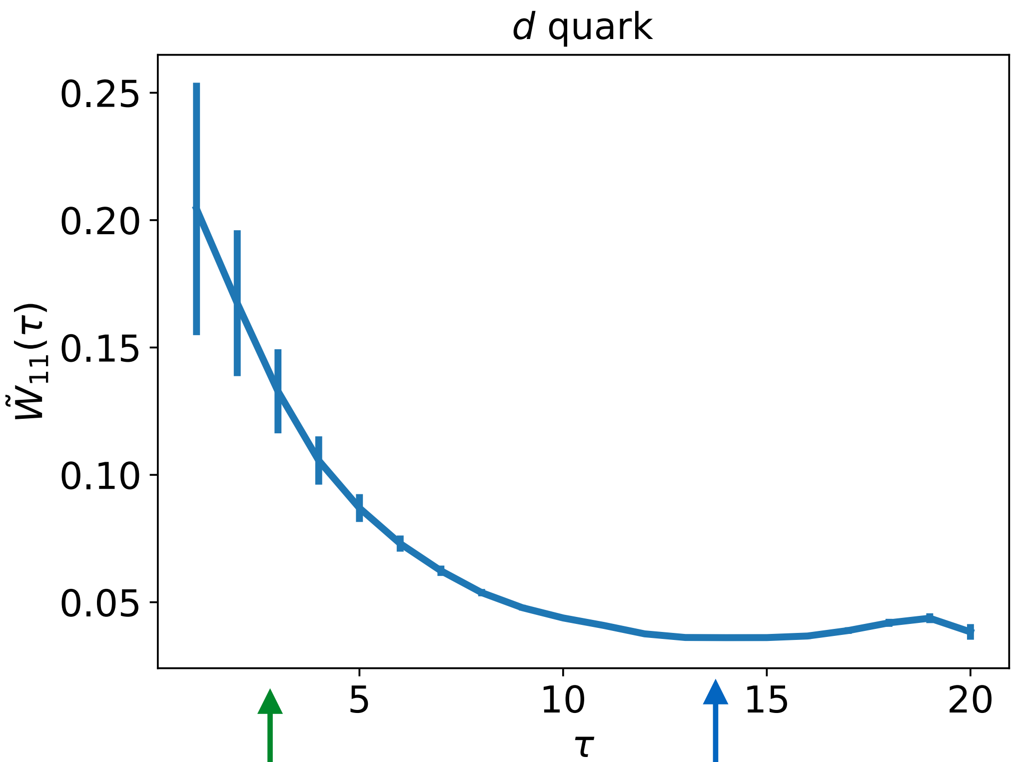
note, different x scale

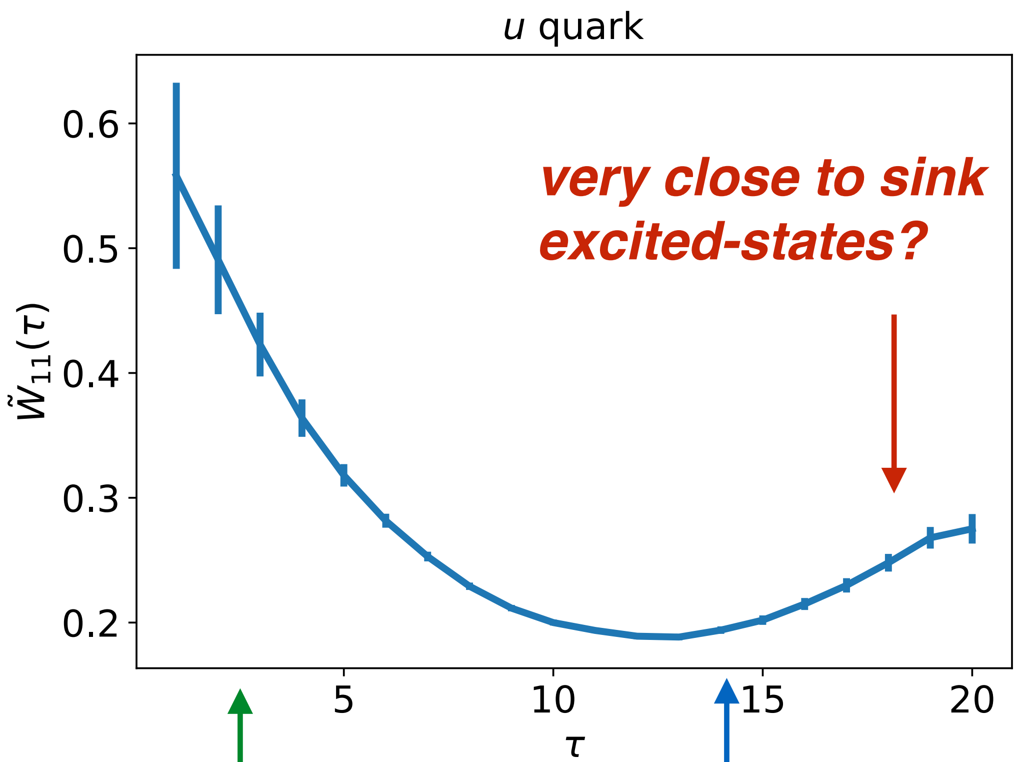
Large momentum transfer

$\boldsymbol{p}$	$\boldsymbol{q}$	E_p	$E_{n=0}$	$\lvert\boldsymbol{q}\rvert$	ν	Q^2	x
(0,3,3)	(0,-6,-6)	2.15	2.15	3.57	[2.96, 3.68]	[4, 2]	[0.16, 0.07]

$\mu = \nu = 1$ and $p_1 = q_1 = 0$ $\quad W_{11}(\nu) = F_1(x, Q^2)$

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \sum_n A_n e^{-(E_n - E_p)\tau} \qquad \boldsymbol{p} + \boldsymbol{q} = -\boldsymbol{p} \qquad E_0 = (m_N^2 + \lvert \boldsymbol{p} + \boldsymbol{q} \rvert^2) = E_p$$





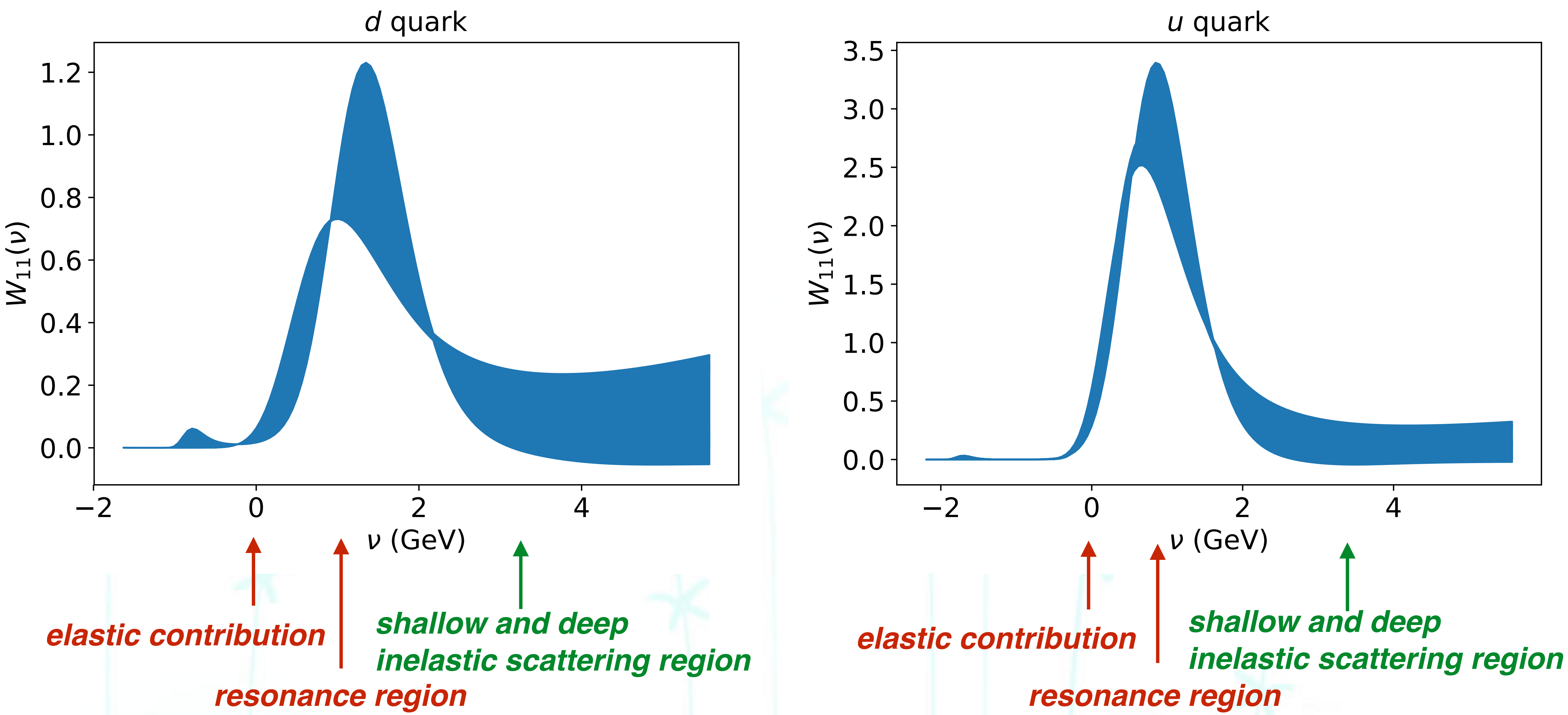
elastic contribution (flat)

higher intermediate-states contribution (exponentially decay)

elastic contribution (flat)

higher intermediate-states contribution (exponentially decay)

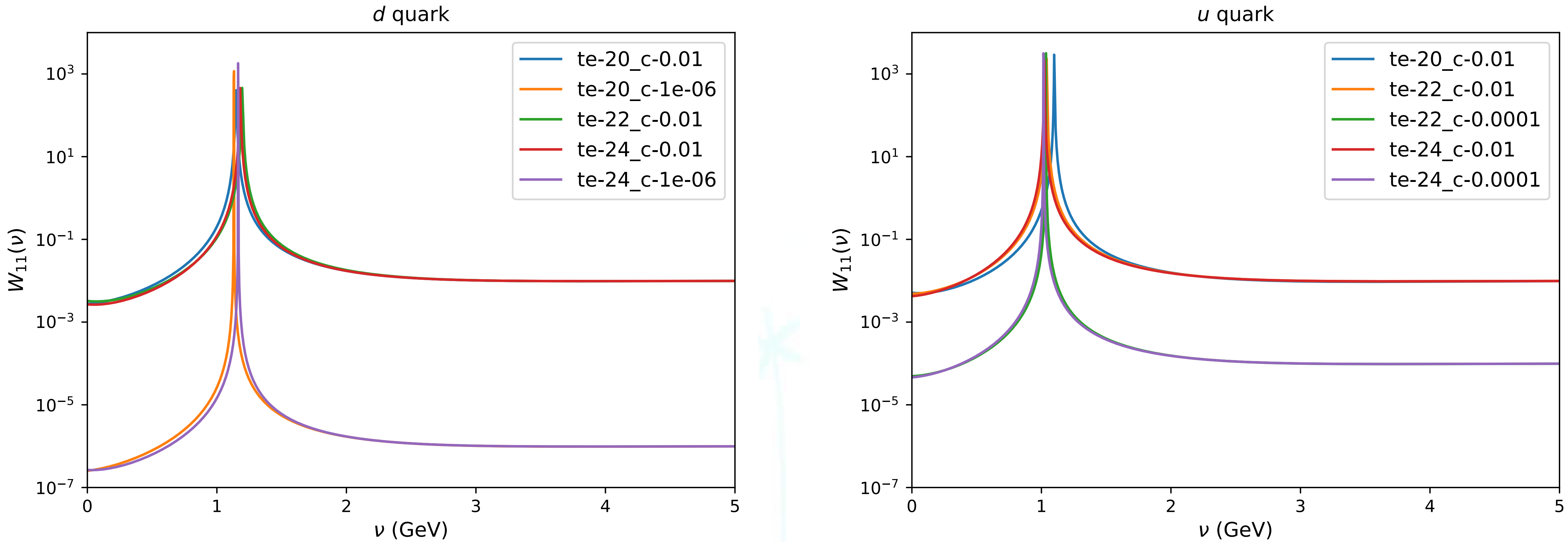
Minkowski hadronic tensor (after ME)



- **Elastic contribution is suppressed by the large momentum transfer.** $G^2(0) \propto \frac{1}{\left(1+\frac{Q_{\mathrm{el}}^2}{\Lambda^2}\right)^4}$

  $Q^2 \sim 13\ \mathrm{GeV}^2,\ G^2(0) \sim 10^{-5}$
- **RES contribution is large and relatively stable.**
- **Large error in the SIS and DIS region, no enough constraint from the data**

How about BR



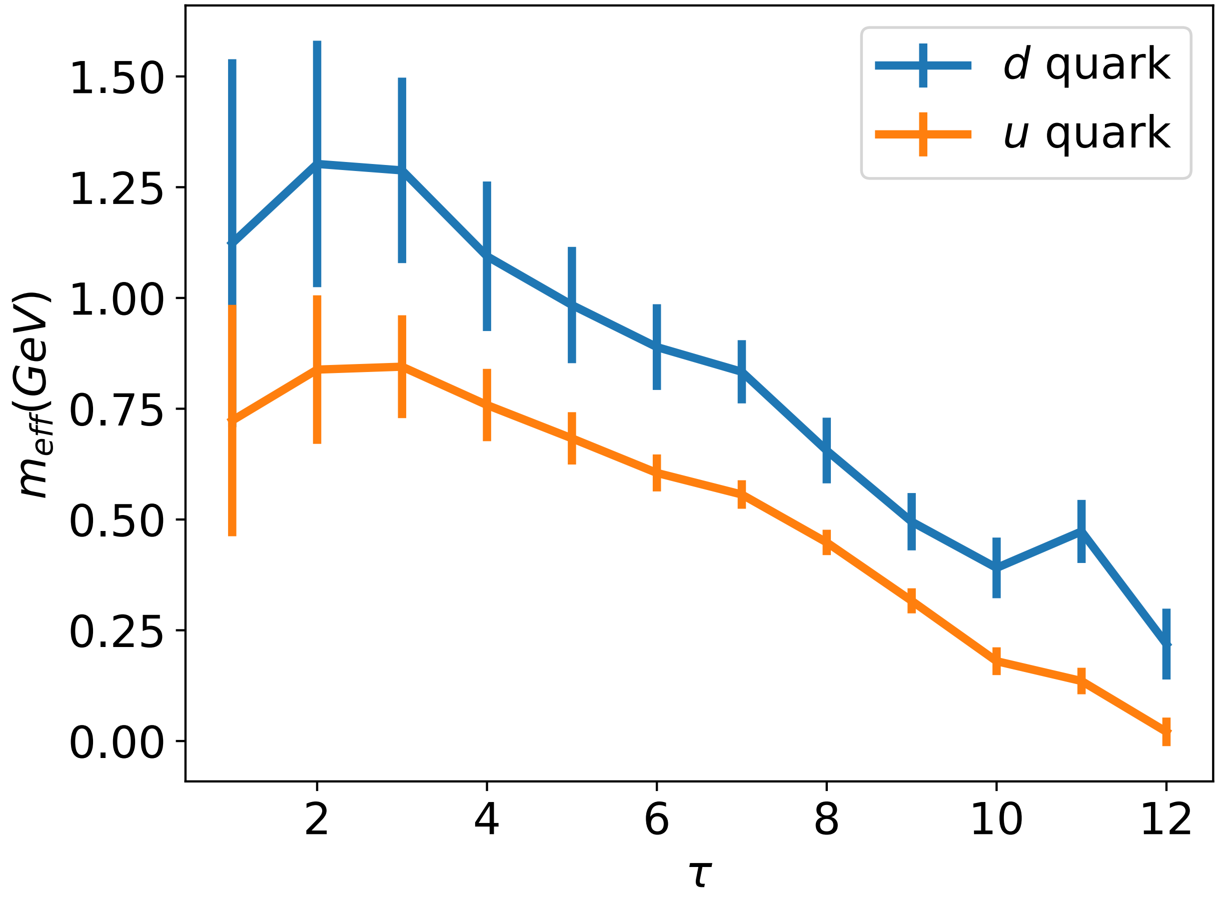
- ✦ **similar structures show around 1 GeV (but shaper)**
- ✦ **quickly approach to the default model after 2 GeV**

Check the effective mass

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \sum_n A_n e^{-(E_n-E_p)\tau} \qquad E_p \sim 2.15 \text{ GeV}$$

one can check the effective mass of the Euclidean hadronic tensor

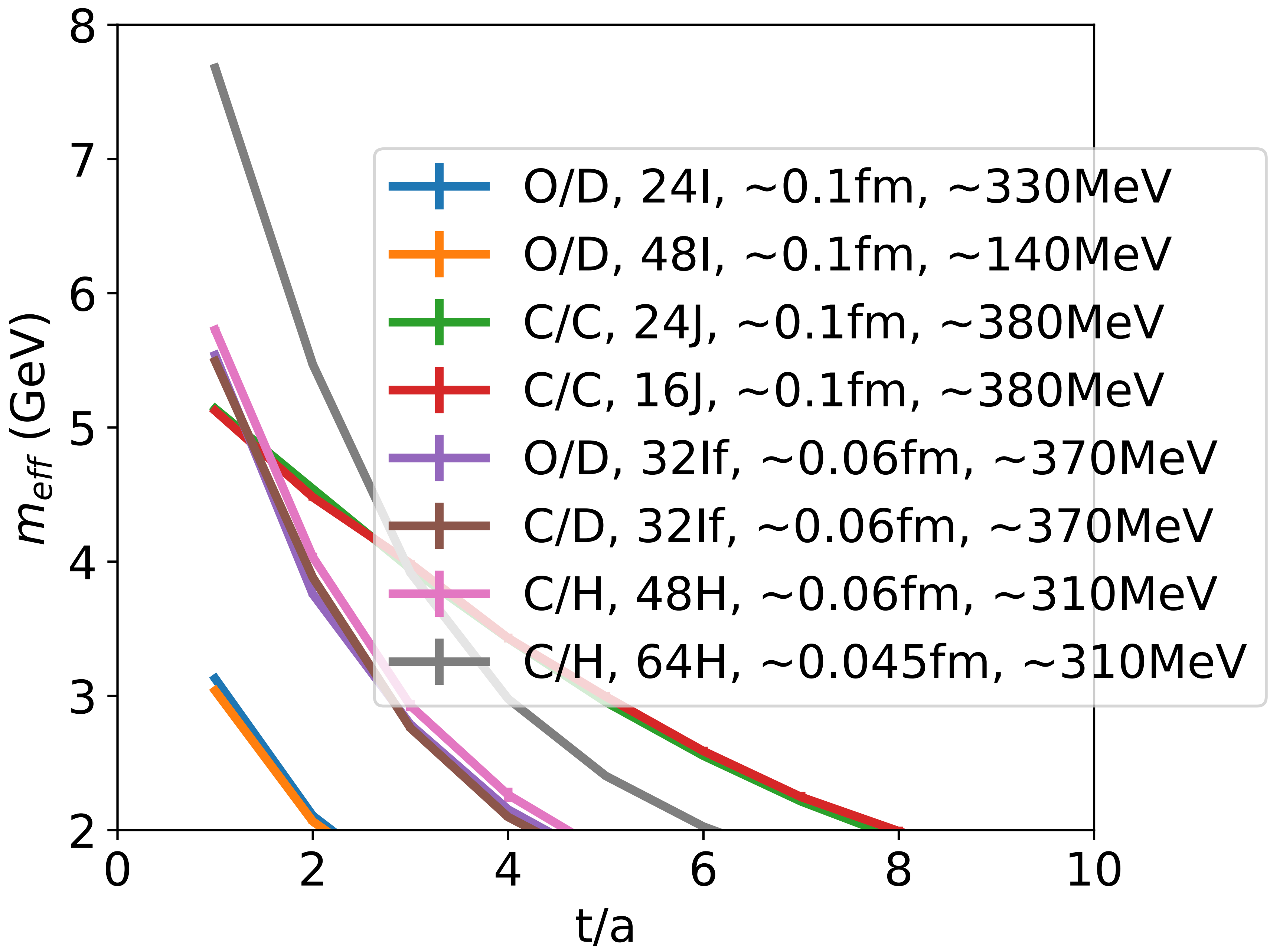
$$m_{\text{eff}} = \log[\tilde{W}(\tau)/\tilde{W}(\tau+1)]$$



$E_n - E_p \sim 1$ GeV $\quad E_n \sim 3.2$ GeV

lattice artifacts: finite volume (resulting in discrete momenta and discrete spectrum)? finite lattice spacing (an UV cutoff)? and/or unphysical pion mass (unphysical multi-particle states)?

Learn more from two-point functions



- ✦ It seems how high we can reach is mainly connected to the lattice spacings.
- ✦ Other factors are not significant.
- ✦ The a~0.045 fm lattice can be a much better choice.

Summary and outlook

- ✦ Calculating the hadronic tensor on the lattice would be helpful to the neutrino experiments and to understand more about the nucleon structure.
- ✦ This might be the only lattice approach that can have inclusive results in the Elastic, RES and SIS region.
- ✦ We can have reasonable results for the elastic contributions.
- ✦ We find that the lattice spacing plays an important role to reach higher excited states.
- ✦ We are working on lattices with smaller lattice spacings to have better results.
- ✦ There are more applications.

Thank you for your attention!

Thank you for your attention!

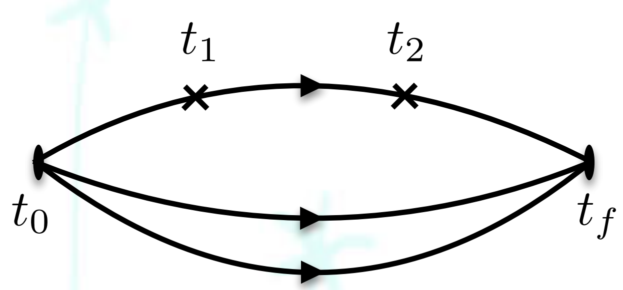
Lattice setups

clover anisotropic lattice, $24^3 \times 128,\ a_t \sim 0.035\ \text{fm},\ m_\pi \sim 380\ \text{MeV},\ \frac{2\pi}{L} \sim 0.42\ \text{GeV}$

H.-W. Lin et al., PRD 79, 034502 (2009)

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F_2(x, Q^2)$$

$\mu = \nu = 1$ and $p_1 = q_1 = 0 \quad W_{11}(\nu) = F_1(x, Q^2)$



two sequential-sources for each 4-point function
554 configurations, 16 source positions

The *x*-range can be reached on this lattice is roughy [0.05, 0.3] by combining different kinematic setups.

This calculation:

$\boldsymbol{p}$	$\boldsymbol{q}$	E_p	$E_{n=0}$	$\lvert\boldsymbol{q}\rvert$	ν	Q^2	x
(0,3,3)	(0,-6,-6)	2.15	2.15	3.57	[2.96, 3.68]	[4, 2]	[0.16, 0.07]

More on the setups

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \sum_n A_n e^{-(E_n - E_p)\tau}$$

energy of the intermediate state n

external nucleon energy

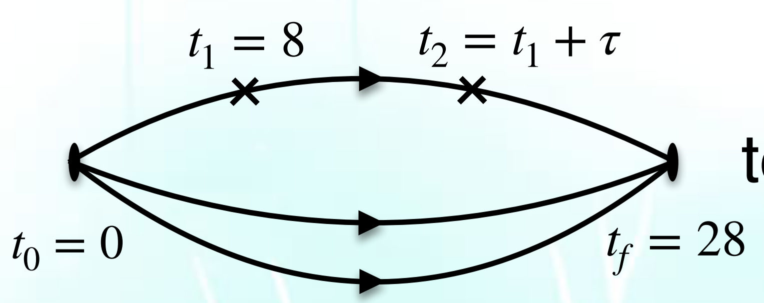
p=(033), ***q***=(0-6-6) $\boldsymbol{p} + \boldsymbol{q} = -\boldsymbol{p}$

$$E_0 = (m_N^2 + |\boldsymbol{p} + \boldsymbol{q}|^2) = E_p$$

the lowest energy of intermediate states

for small τ, higher intermediate states contribute, exponentially decay

for large τ, lowest intermediate state (elastic contribution) dominates, constant

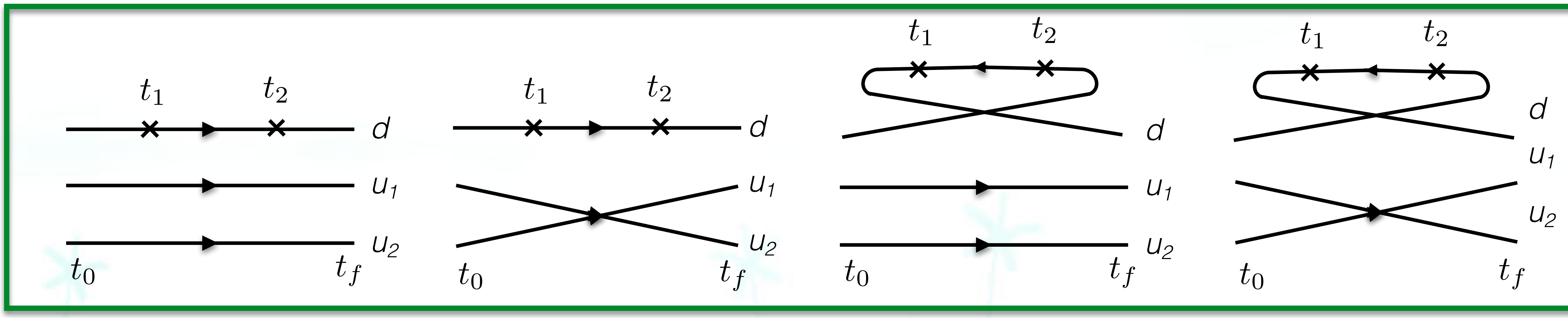


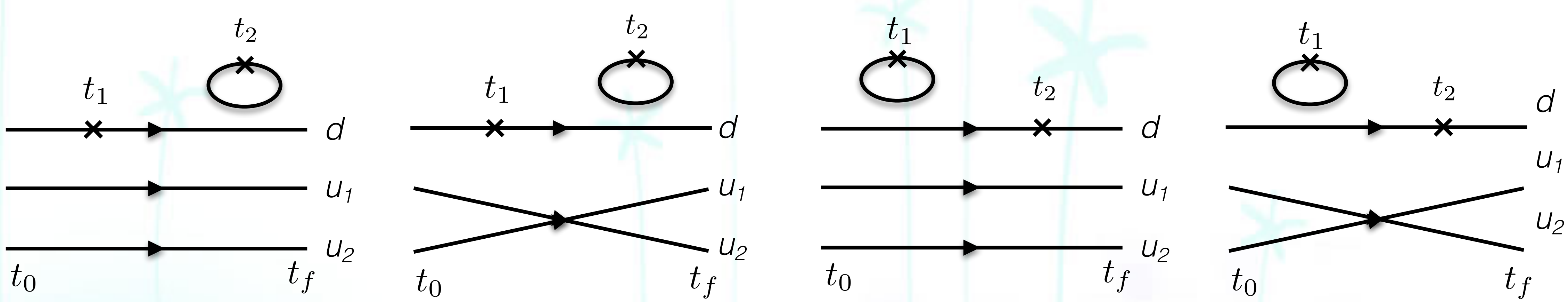
to avoid the contact point and sink excited stats $\tau \in [1,\ 12]$

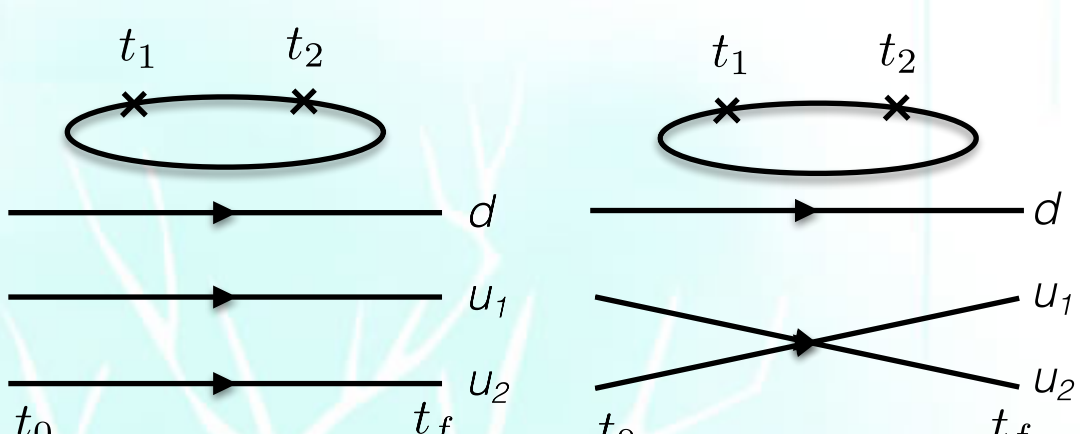
Contractions (d quark)

$$C_4 = \sum_{\boldsymbol{x_f}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x_f}} \sum_{\boldsymbol{x_2}\boldsymbol{x_1}} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C\gamma_5 d\right] u_2$$





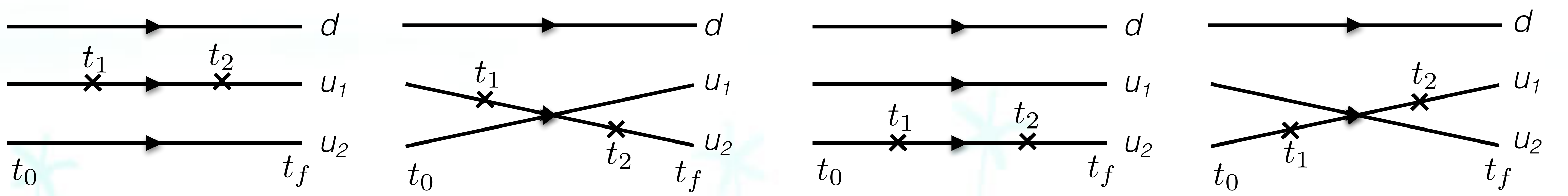


connected insertions only for now

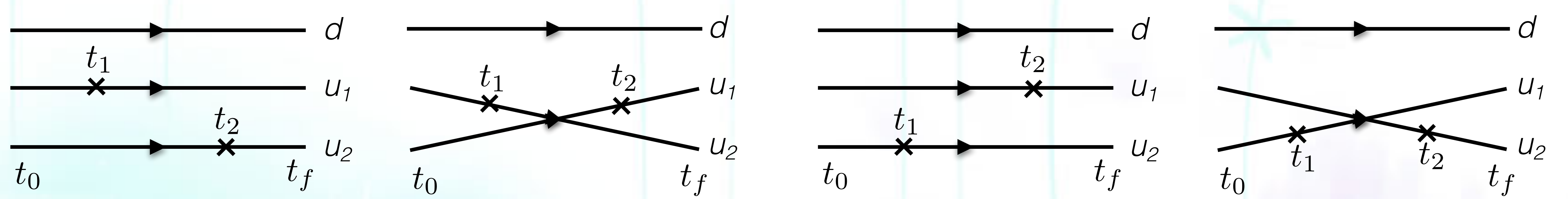
Contractions (*u* quark)

$$C_4 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \sum_{\boldsymbol{x}_2 \boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2 - \boldsymbol{x}_1)} \left\langle \chi_N(\boldsymbol{x}_f, t_f) J_\mu(\boldsymbol{x}_2, t_2) J_\nu(\boldsymbol{x}_1, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C \gamma_5 d\right] u_2$$



plus all possible backward propagating ones



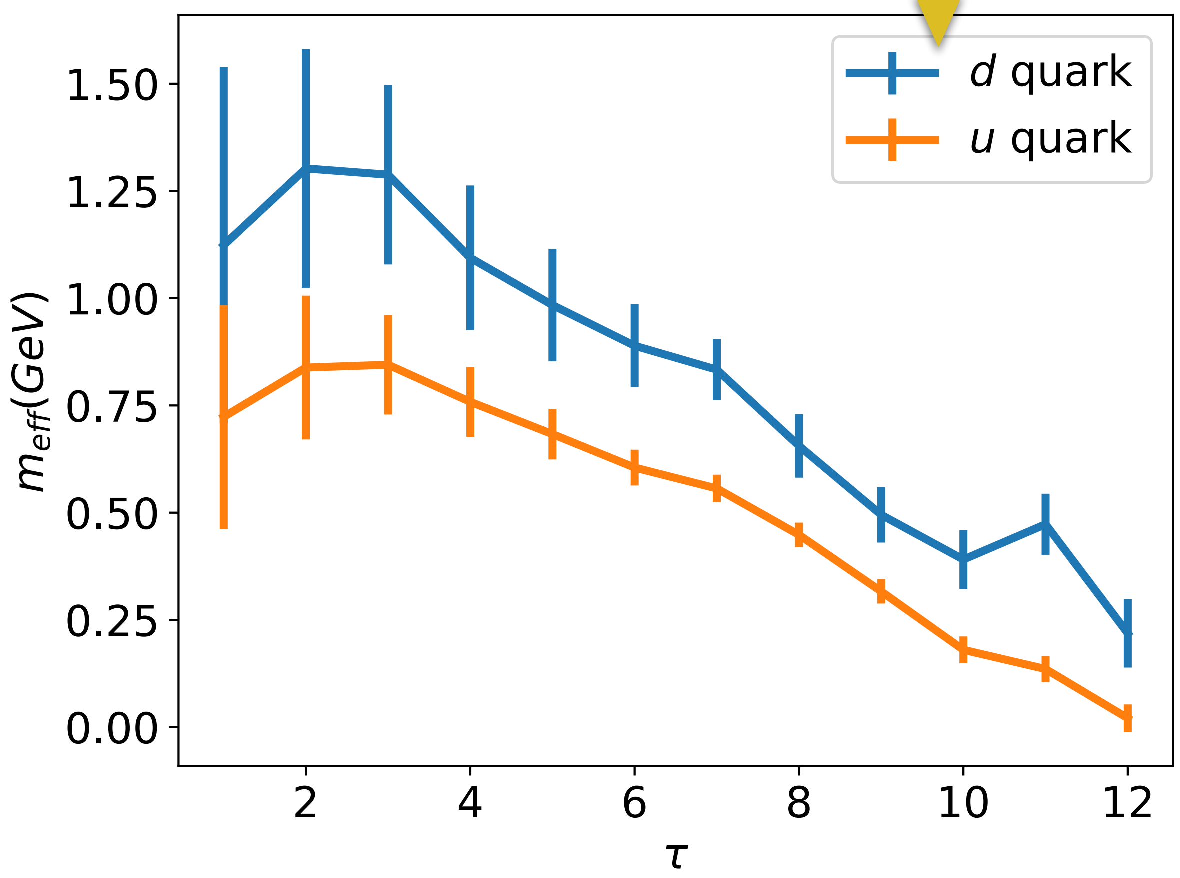
currents can be on two different quark lines respectively (cat ear diagrams)

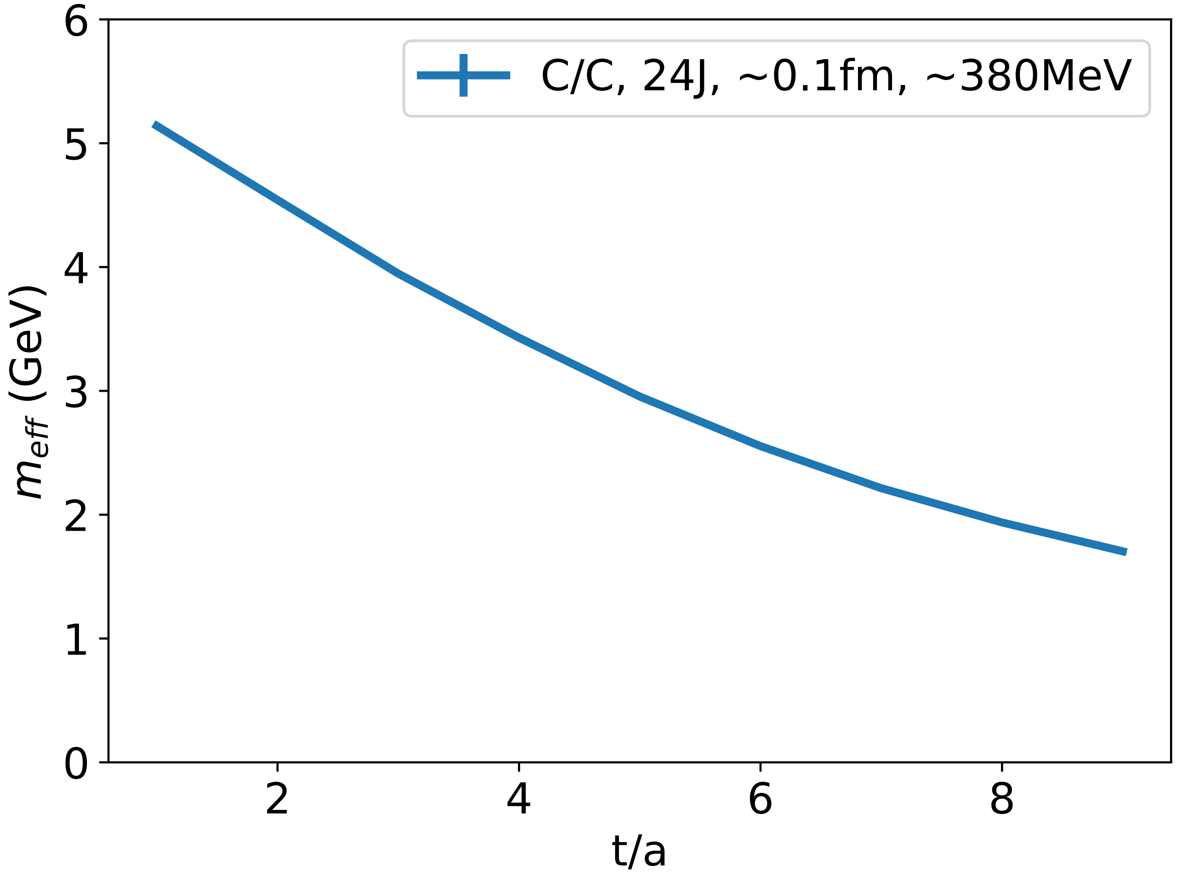
Check the effective mass

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \sum_n A_n e^{-(E_n-E_p)\tau} \qquad E_p \sim 2.15\ \text{GeV}$$

one can check the effective mass of the Euclidean hadronic tensor

$$m_{\text{eff}} = \log[\tilde{W}(\tau)/\tilde{W}(\tau+1)]$$





$E_n - E_p \sim 1$ GeV $\quad E_n \sim 3.2$ GeV

one can also check the effective mass of vector meson with point source